Supplemental material for ocean acoustics

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0.1 The relation between pressure and displacement potential

First introduce the conservation of mass formula

$$\frac{\partial \rho}{\partial t} = -\rho_0(\nabla \cdot \vec{v}). \tag{1}$$

Now let us assume the wave takes a sinusoidal variation, i.e., we have:

$$\vec{v} = \tilde{v}\mathrm{e}^{\mathrm{j}\vec{k}\cdot\vec{r}} = \tilde{v}e^{i(k_x x + k_y y + k_z z)} \tag{2a}$$

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \left(ik_x\tilde{v}_x + ik_y\tilde{v}_y + ik_z\tilde{v}_z\right)e^{i(k_x x + k_y y + k_z z)} = i\vec{k}\cdot\tilde{v}e^{i(k_x x + k_y y + k_z z)} = i\vec{k}\cdot\vec{v} \tag{2b}$$

$$\frac{\partial \rho}{\partial t} = -i\omega\rho \tag{2c}$$

Combing (1) and (2), we have

$$-i\omega\rho = -\rho_0 i\vec{k}\cdot\vec{v}. \tag{3}$$

According to $p/\rho = c^2$, we utilize equation (3) and eliminate ρ, which yields

$$\omega\frac{p}{c^2} = \rho_0\vec{k}\cdot\vec{v}. \tag{4}$$

Because the acoustics wave is a longitudinal wave, the wave vector $\vec{k}$ is of the same direction of particle velocity $\vec{v}$, and $\vec{k}\cdot\vec{v} = kv$. Thus (4) can be simplified further as

$$\omega\frac{p}{c^2} = \rho_0 kv. \tag{5}$$

Note that the sound speed is defined as

$$c \triangleq \frac{\omega}{k}. \tag{6}$$

Substituting (6) into (5), we obtain

$$\frac{p}{v} = \rho_0 c. \tag{7}$$

Next we establish the relationship between the displacement potential and the pressure.

First recall the definition of the displacement potential,

$$\vec{u} \triangleq \nabla\psi \tag{8}$$

From the kinematic equation (the velocity is the time derivative of the displacement)

$$\vec{v} = \frac{\partial \vec{u}}{\partial t} = \frac{\partial}{\partial t}\nabla\psi \tag{9}$$

Substituting (9) into (1), we have

$$\frac{\partial \rho}{\partial t} = -\rho_0\nabla\cdot\frac{\partial}{\partial t}\nabla\psi = -\frac{\partial}{\partial t}\left(\rho_0\nabla^2\psi\right). \tag{10}$$

Recall that $\frac{p}{\rho} = c^2$, equation (10) is further simplified as

$$p = -\rho_0 c^2\nabla^2\psi = -K\nabla^2\psi \tag{11}$$

where

$$K = \rho_0 c^2 \tag{12}$$

. Now we reveal the relationship between pressure and displacement potential.

0.2 The derivation of wave equation

Here we derive the **wave equation**. First we assume ρ is independent of spatial coordinates and the original equations can be simplified as

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \vec{v}, \tag{13a}$$
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p, \tag{13b}$$
$$p = p_0 + \rho' \left[\frac{\partial p}{\partial \rho}\right]_S. \tag{13c}$$

The sound speed is defined as

$$c^2 = \left[\frac{\partial p}{\partial \rho}\right]_S. \tag{14}$$

We decompose the pressure p as

$$p = p_0 + p', \tag{15}$$

where p' is the infinitesimal perturbation of pressure.

Combining (13c) . (14) and (15), we obtain

$$p' = c^2 \rho'. \tag{16}$$

Since p' is the infinitesimal perturbation, $p' \ll p_0$,. By assuming $v \ll c$, we obtain

$$(\vec{v} \cdot \nabla)\vec{v} = 0. \tag{17}$$

According to (17), equations (13a) and (13b) are simplified further as

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \vec{v}. \tag{18a}$$
$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho}\nabla p'. \tag{18b}$$

We take the time derivative of the left side of (18a), and substitute (18b)(Note that the time derivative and ∇ can be interchanged) , we obtain

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 \rho'}{\partial t^2} = -\rho \frac{\partial}{\partial t}\nabla \cdot \vec{v} = -\rho \nabla \cdot \frac{\partial \vec{v}}{\partial t} = -\rho \nabla \cdot \left(-\frac{1}{\rho}\nabla p'\right) = \nabla \cdot \nabla p' = \nabla^2 p'. \tag{19}$$

Utilizing equation (16) and eliminating ρ, we obtain

$$\nabla^2 p' - \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} = 0, \tag{20}$$

where we have assumed c is independent of time.

If we delete primes at ρ, p, equation (20) is simplified further as

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0. \tag{21}$$

0.3 Green's theorem

Here we derive Green's theorem, first we need to proof Green's formula.

Green's Formula: Suppose u($\mathbf{r}$),v($\mathbf{r}$), S is the boundary of V,

Then

$$\iiint_V \left[u(\mathbf{r})\nabla^2 v(\mathbf{r}) - v(\mathbf{r})\nabla^2 u(\mathbf{r})\right] dV = \oint_S \left[u(\mathbf{r})\nabla v(\mathbf{r}) - v(\mathbf{r})\nabla u(\mathbf{r})\right] \cdot d\mathbf{S}. \tag{22}$$

Proof: From the vector indentity

$$\nabla \cdot (u\mathbf{v}) = \nabla u \cdot \mathbf{v} + u\nabla \cdot \mathbf{v}. \tag{23}$$

The integrand of the volume integration of (22) is

$$u(\mathbf{r})\nabla^2 v(\mathbf{r}) - v(\mathbf{r})\nabla^2 u(\mathbf{r}) = \nabla \cdot [u(\mathbf{r})\nabla v(\mathbf{r}) - v(\mathbf{r})\nabla u(\mathbf{r})]\,. \tag{24}$$

Applying Gauss's Formula

$$\iiint_V \nabla \cdot \mathbf{v} dV = \oint_S \mathbf{v} \cdot d\mathbf{S}. \tag{25}$$

We obtained

$$\iiint_V \left[u(\mathbf{r})\nabla^2 v(\mathbf{r}) - v(\mathbf{r})\nabla^2 u(\mathbf{r})\right] dV = \iiint_V \nabla \cdot [u(\mathbf{r})\nabla v(\mathbf{r}) - v(\mathbf{r})\nabla u(\mathbf{r})]\, dV \tag{26a}$$

$$= \oint_S [u(\mathbf{r})\nabla v(\mathbf{r}) - v(\mathbf{r})\nabla u(\mathbf{r})] \cdot d\mathbf{S}. \tag{26b}$$

The proof is over.
If we let

$$u(\mathbf{r}) = G_\omega(\mathbf{r}, \mathbf{r_0}), v(\mathbf{r}) = \psi(\mathbf{r_0}). \tag{27}$$

Applying the Green's formula with integrands as (27)

$$\iiint_V \left[G_\omega(\mathbf{r}, \mathbf{r_0})\nabla^2\psi(\mathbf{r_0}) - \psi(\mathbf{r_0})\nabla^2 G_\omega(\mathbf{r}, \mathbf{r_0})\right] dV_0 = \oint_S [G_\omega(\mathbf{r}, \mathbf{r_0})\nabla\psi(\mathbf{r_0}) - \psi(\mathbf{r_0})\nabla G_\omega(\mathbf{r}, \mathbf{r_0})] \cdot d\mathbf{S_0}. \tag{28a}$$

$$= \oint_S \left[G_\omega(\mathbf{r}, \mathbf{r_0})\frac{\partial\psi(\mathbf{r_0})}{\partial\mathbf{n_0}} - \psi(\mathbf{r_0})\frac{\partial G_\omega(\mathbf{r}, \mathbf{r_0})}{\partial\mathbf{n_0}}\right] dS_0, \tag{28b}$$

which is the **Green's theorem** of boundary value problems.

0.4 axisymmetric problem

Here we show why the right hand of depth-separated wave equation is $S_\omega \dfrac{\delta(z - z_s)}{2\pi}$ in axisymmetric problem after Hankel transform.

First we need to prove the **equivalent relation** between δ function that for a point source at position $(0, 0, z_s)$ in Cartesian system , its force term $S_\omega\delta(x)\delta(y)\delta(z - z_s)$ in cylindrical system is of the form:

$$S_\omega\delta(x)\delta(y)\delta(z - z_s) = \frac{S_\omega}{2\pi r}\delta(r)\delta(z - z_s). \tag{29}$$

proof: We just need to verify that

$$\begin{cases} \iiint_V \dfrac{S_\omega}{2\pi r}\delta(r)\delta(z - z_s)dV = 0, & (0, 0, z_s) \notin V \\ \iiint_V \dfrac{S_\omega}{2\pi r}\delta(r)\delta(z - z_s)dV = S_\omega, & (0, 0, z_s) \notin V \end{cases}$$

Here $(0, 0, z_s)$ is Cartesian coordinate of the point source. This is easily to be verified (if you are interested you can dig into it).

Therefore the Helmholtz equation for axisymmetric propagation problem (a point source) in cylindrical system is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = S_w\frac{\delta(r)\delta(z - z_s)}{2\pi r}. \tag{31}$$

Applying Hankel transform

$$\psi(k_r, z) = \int_0^\infty \psi(r, z)J_0(k_r r)rdr. \tag{32}$$

to (31), we will obtain

$$\int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)J_0(k_r r)rdr + \int_0^\infty \frac{\partial^2\psi}{\partial z^2}J_0(k_r r)rdr + k^2\psi(k_r,z) = \int_0^\infty S_w\frac{\delta(r)\delta(z-z_s)}{2\pi r}J_0(k_r r)rdr. \tag{33}$$

Next we simplify equation (33) to obtain *depth-separated wave equation* (42) below.

We know

$$J_0(0) = 1. \tag{34}$$

, so the right hand side of (33) is

$$\int_0^\infty S_w\frac{\delta(r)\delta(z-z_s)}{2\pi r}J_0(k_r r)rdr = \frac{S_\omega}{2\pi}\delta(z-z_s). \tag{35}$$

The second term of (33) is

$$\int_0^\infty \frac{\partial^2\psi(r,z)}{\partial z^2}J_0(k_r r)rdr = \frac{d^2}{dz^2}\psi(k_r,z). \tag{36}$$

Then we need to compute the first term $\int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)J_0(k_r r)rdr$ using the regular method as we prove the properties of Fourier transform as before (signal and systems).

Computation:

Recall the properties of Bassel function of ν order:

$$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x). \tag{37a}$$

$$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x). \tag{37b}$$

For a function $\psi(r,z)$ whose Hankel transform is $\psi(k_r,z)$, the inverse transform is

$$\psi(r,z) = \int_0^\infty \psi(k_r,z)J_0(k_r r)k_r dk_r. \tag{38}$$

Applying partial derivation to $\psi(r,z)$ with respect to r and use (37b), we obtain

$$\frac{\partial\psi(r,z)}{\partial r} = \int_0^\infty \psi(k_r,z)\frac{\partial}{\partial k_r r}\left(J_0(k_r r)\right)k_r^2 dk_r = -\int_0^\infty \psi(k_r,z)J_1(k_r r)k_r^2 dk_r. \tag{39}$$

Make use of (37a) and compute

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi(r,z)}{\partial r}\right) = -\int_0^\infty \psi(k_r,z)\frac{1}{r}\frac{\partial}{\partial k_r r}\left(k_r r J_1(k_r r)\right)k_r^2 dk_r = -\int_0^\infty k_r^2\psi(k_r,z)J_0(k_r r)k_r dk_r. \tag{40}$$

Therefore $-k_r^2\psi(k_r,z)$ is the Hankel transform of $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi(r,z)}{\partial r}\right)$, in other words , the first term in (33) is

$$\int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)J_0(k_r r)rdr = -k_r^2\psi(k_r,z). \tag{41}$$

Substituting (35), (36) and (41) to (33), we obtain

$$\frac{d^2}{dz^2}\psi + (k^2 - k_r^2)\psi = \frac{S_\omega}{2\pi}\delta(z-z_s). \tag{42}$$

0.5 Wavenumber integration and solution for ideal waveguide

One of the properties of the ν order Hankel function is

$$H_\nu^{(2)}(z\mathrm{e}^{jm\pi}) = \frac{\sin\,(1+m)\nu\pi}{\sin\nu\pi} H_\nu^{(2)}(z) + \mathrm{e}^{\nu\pi j}\frac{\sin m\nu\pi}{\sin\nu\pi} H_\nu^{(1)}(z). \tag{43}$$

Proof:
Because

$$J_\nu(z\mathrm{e}^{jm\pi}) = \mathrm{e}^{jm\nu\pi} J_\nu(z). \tag{44a}$$

$$J_{-\nu}(z\mathrm{e}^{jm\pi}) = \mathrm{e}^{-jm\nu\pi} J_{-\nu}(z). \tag{44b}$$

And the *Neumann function* $Y_\nu(z)$ is the linear combination of Bessel functions

$$Y_\nu(z) = \frac{\cos\nu\pi J_\nu(z) - J_{-\nu}(z)}{\sin\nu\pi}. \tag{45}$$

In equation (45), we let $z = z\mathrm{e}^{jm\pi}$, combining (44a) and (44b)

$$Y_\nu(z\mathrm{e}^{jm\pi}) = \frac{\cos\nu\pi\mathrm{e}^{jm\nu\pi} J_\nu(z) - \mathrm{e}^{-jm\nu\pi} J_{-\nu}(z)}{\sin\nu\pi}. \tag{46}$$

According (45)

$$J_{-\nu}(z) = \cos\nu\pi J_\nu(z) - \sin\nu\pi Y_\nu(z) \tag{47}$$

We utilize equation (47) in (46), equation (46) is further simplified as

$$Y_\nu(ze^{im\pi}) = \frac{2j\sin m\nu\pi\cos\nu\pi J_\nu(z) + \mathrm{e}^{-jm\nu\pi}\sin\nu\pi Y_\nu(z)}{\sin\nu\pi}. \tag{48}$$

From the definition of the Hankel function

$$H_\nu^{(2)}(z\mathrm{e}^{jm\pi}) = J_\nu(z\mathrm{e}^{jm\pi}) - jY_\nu(z\mathrm{e}^{jm\pi}). \tag{49}$$

From the definition of Hankel functions:

$$J_\nu(z) = \frac{1}{2}(H_\nu^{(1)}(z) + H_\nu^{(2)}(z)) \tag{50a}$$

$$Y_\nu(z) = \frac{1}{2j}(H_\nu^{(1)}(z) - H_\nu^{(2)}(z)) \tag{50b}$$

We combine equations (44a) (48) (50a) and (50b) in (49), we get (43). The proof is over.
If we let $m = -1$ in (43) ,then we have

$$H_\nu^{(2)}(z\mathrm{e}^{-j\pi}) = -\mathrm{e}^{\nu\pi j} H_\nu^{(1)}(z). \tag{51}$$

Moreover, if we let $\nu = 0$ in (51) ,we obtain

$$H_0^{(2)}(z\mathrm{e}^{-j\pi}) = -H_0^{(1)}(z). \tag{52}$$

The wavenumber integral (textbook section **2.4.3.3** (2.131)) is

$$
\begin{aligned}
\psi(r,z) &= \int_0^\infty A_1^-(k_r)\mathrm{e}^{-jk_{z,1}z}J_0(k_r r)k_r dk_r \\
&= \int_0^\infty A_1^-(k_r)\mathrm{e}^{-jk_{z,1}z}\frac{H_0^{(1)}(k_r r)+H_0^{(2)}(k_r r)}{2}k_r dk_r \\
&= \frac{1}{2}\int_0^\infty A_1^-(k_r)\mathrm{e}^{-jk_{z,1}z}H_0^{(1)}(k_r r)k_r dk_r + \frac{1}{2}\int_0^\infty A_1^-(k_r)\mathrm{e}^{-jk_{z,1}z}H_0^{(2)}(k_r r)k_r dk_r \\
&= \frac{1}{2}\int_0^\infty A_1^-(k_r)\mathrm{e}^{-jk_{z,1}z}H_0^{(1)}(k_r r)k_r dk_r + \frac{1}{2}\int_0^{-\infty} A_1^-(k_r)\mathrm{e}^{-jk_{z,1}z}H_0^{(2)}(k_r r\mathrm{e}^{-j\pi})k_r dk_r \\
&use(52) \\
&= \frac{1}{2}\int_0^\infty A_1^-(k_r)\mathrm{e}^{-jk_{z,1}z}H_0^{(1)}(k_r r)k_r dk_r + \frac{1}{2}\int_{-\infty}^0 A_1^-(k_r)\mathrm{e}^{-jk_{z,1}z}H_0^{(1)}(k_r r)k_r dk_r \\
&= \frac{1}{2}\int_{-\infty}^\infty A_1^-(k_r)\mathrm{e}^{-jk_{z,1}z}H_0^{(1)}(k_r r)k_r dk_r.
\end{aligned}
\tag{53}
$$

Normal Modes solution (section 2.4.4.3) by calculating residues
From the wavenumber integration (53) (Let the kernel of intergration be $\psi(k_r, z)$)

$$
\psi(r,z) = \frac{1}{2}\int_{-\infty}^\infty \psi(k_r,z)H_0^{(1)}(k_r r)k_r dk_r. \tag{54}
$$

And the solution of depth-searated wave equation for ideal waveguide is:

$$
\psi(k_r,z) = -\frac{S_\omega}{2\pi}\begin{cases}\dfrac{\sin k_z z\sin k_z(D-z_s)}{k_z\sin k_z D}, & z<z_s \\ \dfrac{\sin k_z z_s\sin k_z(D-z)}{k_z\sin k_z D}, & z>z_s\end{cases} \tag{55}
$$

Computation of integration (54):
Using the residue theorem

$$
\psi(r,z) = \pi i\sum_{m=1}^\infty Res\left[\psi(k_r,z)H_0^{(1)}(k_r r)k_r\right]_{k=k_{rm}}. \tag{56}
$$

Because the poles of integrand is of first order, the residue is computed by

$$
Res\left[\psi(k_r,z)H_0^{(1)}(k_r r)k_r\right]|_{k=k_{rm}} = \lim_{k_r\to k_{rm}}(k_r-k_{rm})\psi(k_r,z)H_0^{(1)}(k_r r)k_r. \tag{57}
$$

With

$$
k_{rm} = \sqrt{k^2-\left(\frac{m\pi}{D}\right)^2}. \tag{58a}
$$

$$
k_{zm} = \frac{m\pi}{D}. \tag{58b}
$$

The poles occur at function $\psi(k_r, z)$, thus

$$
\begin{aligned}
Res\left[\psi(k_r,z)H_0^{(1)}(k_r r)k_r\right]|_{k=k_{rm}} &= \lim_{k_r\to k_{rm}}(k_r-k_{rm})\psi(k_r,z)H_0^{(1)}(k_r r)k_r \\
&= \lim_{k_r\to k_{rm}}(k_r-k_{rm})\psi(k_r,z)*H_0^{(1)}(k_{rm} r)k_{rm}.
\end{aligned}
\tag{59}
$$

So we just need to compute $\lim_{k_r\to k_{rm}}(k_r-k_{rm})\psi(k_r,z)$, in detail

$$
\lim_{k_r\to k_{rm}}(k_r-k_{rm})\psi(k_r,z) = -\frac{S_\omega}{2\pi}\begin{cases}\lim\limits_{k_r\to k_{rm}}(k_r-k_{rm})\dfrac{\sin k_z z\sin k_z(D-z_s)}{k_z\sin k_z D}, & z<z_s \\ \lim\limits_{k_r\to k_{rm}}(k_r-k_{rm})\dfrac{\sin k_z z_s\sin k_z(D-z)}{k_z\sin k_z D}, & z>z_s\end{cases} \tag{60}
$$

We simplify equation (60) further

$$\begin{aligned}
&\lim_{k_r\to k_{rm}}(k_r-k_{rm})\frac{\sin k_z z\sin k_z(D-z_s)}{k_z\sin k_z D}=\lim_{k_r\to k_{rm}}(k_r-k_{rm})\frac{\sin k_z z_s\sin k_z(D-z)}{k_z\sin k_z D}\\
&=-\frac{\cos m\pi\sin k_{zm}z_s\sin k_{zm}z}{k_{zm}}\lim_{k_r\to k_{rm}}\frac{(k_r-k_{rm})}{\sin k_z D}\\
&=-\frac{\cos m\pi\sin k_{zm}z_s\sin k_{zm}z}{k_{zm}}\lim_{k_r\to k_{rm}}\frac{k_r-k_{rm}}{\sin\sqrt{k^2-k_r^2}D}\\
&=-\frac{\cos m\pi\sin k_{zm}z_s\sin k_{zm}z}{k_{zm}}\lim_{k_r\to k_{rm}}\frac{1}{\cos\sqrt{k^2-k_r^2}D(\frac{-k_rD}{\sqrt{k^2-k_r^2}})}\\
&=\frac{\sin k_{zm}z\sin k_{zm}z_s}{k_{rm}D}.
\end{aligned}\tag{61}$$

where we have used **L'Hospital rule** to obtain the limit.
Therefore equation (60) becomes

$$\lim_{k_r\to k_{rm}}(k_r-k_{rm})\psi(k_r,z)=-\frac{S_\omega\sin k_{zm}z\sin k_{zm}z_s}{2\pi k_{rm}D}.\tag{62}$$

We substitute (62) in (59), the residue is

$$Res\left[\psi(k_r,z)H_0^{(1)}(k_rr)k_r\right]|_{k=k_{rm}}=-\frac{S_\omega\sin k_{zm}z\sin k_{zm}z_s}{2\pi D}H_0^{(1)}(k_{rm}r).\tag{63}$$

Finally we substitute (63) in (56)

$$\psi(r,z)=-\frac{S_\omega i}{2D}\sum_{m=1}^{\infty}\sin k_{zm}z\sin k_{zm}z_sH_0^{(1)}(k_{rm}r).\tag{64}$$