Summary of Problem 4

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I. Summary

Assume a horizontally stratified environment shown in Fig. 1.The source lies in $(r_s, z_s) = (0, 35)$, the receiver lies in $(r_r, z_r) = (0, 25)$. There are $N_{\text{layer}} = 30$ layers. From Fig. 1, one can see that there are $N_{\text{layer}} - 1 = 29$ interfaces. For the first interface (the layer between one and two), its position is $z = 0$. Let z_i denote the position (depth) of the bottom of the ith layer. In the problem, we set $z_i = (i-1)d_z$, $i = 1, \cdots, N_{\text{layer}} - 1$, where $d_z = 5$. For the top $N_{\text{layer}} - 1$ layers, we set the sound speed $c_i = 1500m/s$, the density $\rho_i = 1g/cm^3$, $i = 1, \cdots, N_{\text{layer}} - 1$. For the last layer, we set $z_{N_{\text{layer}}} = N_{\text{layer}} d_z$. And $c_{N_{\text{layer}}} = 1800m/s$, $\rho_{N_{\text{layer}}} = 1.8g/cm^3$. The source strength and the pressure are

$$S_\omega = -\frac{4\pi}{\rho\omega^2}, \tag{1}$$

$$p(r) = \rho\omega^2\psi(\omega, r), \tag{2}$$

where $\psi(\omega, r)$ is the displacement potential.

We define $\mathbf{v}_m(k_r, z_m)$

$$\mathbf{v}_m(k_r, z_m) = \begin{bmatrix} w(k_r, z) \\ \sigma_{zz}(k_r, z) \end{bmatrix}, \quad \text{the bottom of the m the layer or the top of the m} - 1 \text{ th layer} \tag{3}$$

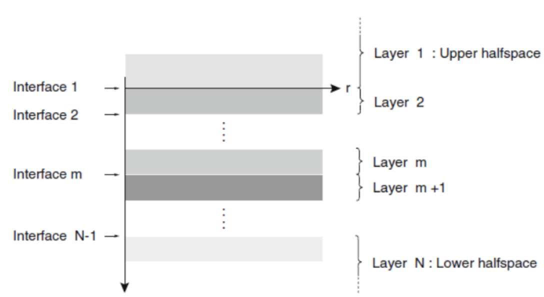

Fig. 1: Horizontally stratified environment

which contains the displacements and stresses at interface m. Similarly, define the degree-of-freedom vector $\mathbf{a}_m(k_r)$ for layer number m as composed of the complex amplitudes of the downgoing and upgoing solutions

$$\mathbf{a}_m(k_r) = \begin{bmatrix} A_m^-(k_r) \\ A_m^+(k_r) \end{bmatrix}, \tag{4}$$

define the local coefficient matrix $\mathbf{c}_m(k_r, z)$

$$\mathbf{c}_m(k_r, z) = \begin{bmatrix} -\mathrm{i}k_z \mathrm{e}^{-\mathrm{i}k_z} & \mathrm{i}k_z \mathrm{e}^{\mathrm{i}k_z} \\ -\rho_m \omega^2 \mathrm{e}^{-\mathrm{i}k_z} & -\rho_m \omega^2 \mathrm{e}^{\mathrm{i}k_z} \end{bmatrix}. \tag{5}$$

We know that

$$\mathbf{v}_m(k_r, z_m) = \mathbf{c}_m(k_r, z_m)\mathbf{a}_m(k_r), \quad \mathbf{v}_m(k_r, z_{m-1}) = \mathbf{c}_m(k_r, z_{m-1})\mathbf{a}_m(k_r). \tag{6}$$

The above relationship reveals that

$$\mathbf{v}_m(k_r, z_{m-1}) = \mathbf{P}_m(k_r)\mathbf{v}_m(k_r, z_m) = \mathbf{c}_m(k_r, z_{m-1})[\mathbf{c}_m(k_r, z_m)]^{-1}\mathbf{v}_m(k_r, z_m). \tag{7}$$

Using the continuity of the field parameters at the interfaces, we can establish a matrix relation between the field parameters at some interface m and the parameters at a lower interface n ($n < m$)

$$\mathbf{v}_n(k_r, z_n) = \mathbf{P}_{n+1}(k_r)\mathbf{v}_{n+1}(k_r, z_{n+1}) = \prod_{i=n+1}^{m} \mathbf{P}_i(k_r)\mathbf{v}_m(k_r, z_m). \tag{8}$$

Since

$$\sigma_{zz}(k_r, z_{N-1}) = -\frac{\rho_N \omega^2}{\mathrm{i}k_{z,N}} w(k_r, z_{N-1}), \tag{9}$$

we rewrite $\mathbf{v}_{N-1}(k_r, z_{N-1})$ as

$$\mathbf{v}_{N-1}(k_r, z_{N-1}) = \begin{bmatrix} w(k_r, z_{N-1}) \\ \sigma_{zz}(k_r, z_{N-1}) \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{\rho_N \omega^2}{\mathrm{i}k_{z,N}} \end{bmatrix} w(k_r, z_{N-1}), \tag{10}$$

Set $w(k_r, z_{N-1}) = 1$ (we will solve $w(k_r, z_{N-1})$ later), and calculate $\mathbf{v}_r(k_r, z_r)$ and $\mathbf{v}_s(k_r, z_s)$ at the receiver and source interface.

Calculate the uppermost surface $\mathbf{v}_1(k_r, z_1)$ as

$$\mathbf{v}_1(k_r, z_1) = \prod_{i=2}^{s} \mathbf{P}_i(k_r)[\prod_{i=s+1}^{N-1} \mathbf{P}_i(k_r)\mathbf{v}_N(k_r, z_{N-1}) + \hat{\mathbf{v}}(k_r, z_s)]$$

$$= \begin{bmatrix} \\ \sigma_{zz}(k_r, z_1) \end{bmatrix} w(k_r, z_{N-1}) + \begin{bmatrix} \\ \hat{\sigma}_{zz}(k_r, z_1) \end{bmatrix} = \begin{bmatrix} \\ 0 \end{bmatrix}, \tag{11}$$

where

$$\hat{\mathbf{v}}(k_r, z_s) = \begin{bmatrix} -\frac{S_\omega}{2\pi} \\ 0 \end{bmatrix}. \tag{12}$$

We obtain

$$w(k_r, z_{N-1}) = -\frac{\hat{\sigma}_{zz}(k_r, z_1)}{\sigma_{zz}(k_r, z_1)} \tag{13}$$

via (11). Knowing $\mathbf{v}(k_r, z)$, using the Hankel transform for each component of $\mathbf{v}(k_r, z)$

$$f(r, z) = \int_0^\infty f(k_r, z) J_0(k_r r) k_r \mathrm{d}k_r, \tag{14}$$

we obtain $\mathbf{v}(r, z)$.

References

[1] "Computational Ocean Accoustics,"