# Principles of Underwater Acoustics, December 2016

# Class discussion

1) **From the Wave Equation to the Helmholtz Equation. Three times. Reading: COA 2.1, 2.2**

The wave equation has some intuitive properties. Think about a 1-d wave traveling down a rope that's being wiggled sinusoidally at one end. If you plotted the displacement vs. time at a fixed location on the string, what would it look like? What would the shape of the string look like at a fixed point in time over all space (think of a flash photograph in a dark room). In both cases, it's a sinusoid. The wave equation is: (double derivative in time of some function) = a constant times (double derivative in space of the same function). Take two derivatives of a sin, and you get -sin. Which is why plotting a wave at a fixed point in space vs time gives a similar plot to plotting something in space at a fixed time.

But usually the first thing that's done to solve the wave equation is to apply an integral transform, turning it into the Helmholtz Equation. Not only does this step get rid of one dimension of the problem, it also tends to get rid of any intuition that you had about the wave equation. Once the intuition is lost, all mathematical manipulations following the Helmholtz Equation then seem like you're just playing with math, without knowing how you got there. Furthermore, "Integral Transform" is a scary phrase. We are going to go from the Wave Equation to the Helmholtz Equation three times, each time getting successively more sophisticated.

In all three cases, you're going to start with the wave equation for displacement potential in COA, Eq 2.26, letting f (r, t) = 0 which just means there is no source present. You will end with Eq 2.26, which is the Helmholtz Equation. In Eq. 2.26, $\psi$ is the displacement potential. $\psi$ of course is a function of space (cars sound quieter when you run away from them) and time (cars sound quieter when their engines are turned off). $\psi = \psi(r, t)$ where r is a vector representing a point in space and t is time.

1. **The Easiest (though not necessarily easy) way.** Assume $\psi(r,t)$ is of the form $\psi(r,\omega)e^{-i\omega t}$. Think about what that means for a moment, and then plug it into Eq 2.26 (remembering to set f(r,t) = 0). Notice that you can take the temporal derivative now that we've assumed a specific form of $\psi(r,t)$ . Take the temporal derivatives, and then manipulate the equation to look like Eq 2.29, remembering the relation between c, k, and $\omega$. Also, because c can be a function of space (the ocean is not homogeneous), k can be a function of r.

2. **The Fourier way.** Write down Eq 2.26, (remembering to set f(r,t) = 0). Now take the Fourier transform of each side. The right hand side remains zero. For the left hand side, we do not know the Fourier transform of $\psi(r, t)$. But whatever it is, we can just call it $\psi(r,\omega)$. We can also switch the order of integral and derivative (in this case) on the right hand side. Manipulate the equation and you should be able to get it to look like Eq 2.29.

3. **The Separation of Variables Way.** To implement separation of variables, we assume that

$\psi(r,t) = R(r)T(t)$. That is, we assume that $\psi(r, t)$ can be written as some function of space R(r) multiplied by some function of time T(t). Plug that into equation 2.26 (remembering to set f(r,t) = 0). Put everything having to do with T(t) on one side of the equation, and everything having to do with R(r) on the other side. Now comes the "trick" in this technique. If two functions of two different variables equal each other, (lets use f(x) = g(y) as a toy example), then the only way the equation can be true for all values of x and y is that f(x) = g(y) = a constant. (Google for "separation of variables", or look in a partial differential equations textbook if you don't believe me). So now we have two equations instead of one. In our toy example, f(x) = a constant, and g(y) = the same constant. In the case of our wave equation, lets call the constant $-k^2$. Of the two equations (one with all of the T(t) and one with all of the R(r), one equation is the Helmholtz equation. The other is a second order linear ordinary differential equation, which has solutions that are complex exponentials. You don't have to solve these two equations – just indicate which one is the Helmholtz one and try to make it look like Eq 2.29.

## 2) Reading: Wikipedia

Soon you will hear phrases like: "Hankel Function", "Bessel Function", "Integral Transform" and "Hankel Transform". Those are scary sounding phrases. So lets try gently easing into them. Go to wikipedia.com and **read the following articles: 1) Bessel Function, 2) Integral Transform,** 3) Hankel Transform. Remember that if someone were to ask "What is a sine function?" you can reply by saying that it is the solution to the differential equation $x''(t) + x(t) = 0$. That is a legitimate definition of the sine function. In some sense, a Bessel function is no different than a sine or cosine – it is simply the solution to some differential equation. Don't worry too much about the various properties of Bessel or Hankel function, just get a general idea for what they are and what they look like.

## 3) Simple Problems

a) You have invented an apparatus that can fry an egg underwater using N dB in certain time interval. How many dB would it take to fry two eggs in the same time ?
b) A ray is launched at 30 deg grazing angle from the surface, assuming a linear sound speed with 1500 m/s at the top and 1450 m/s at the bottom. What would the grazing angle be at the bottom?