Ocean Acoustic Theory

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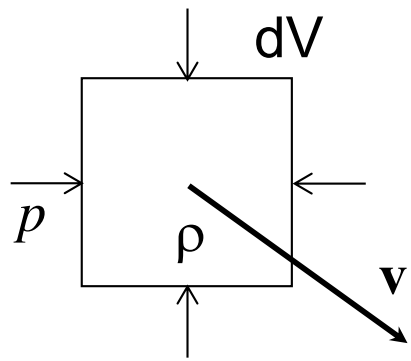
The Acoustic Wave Equation



Conservation of Mass

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v}$$

Euler's Equation (Equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p(\rho)$$

Constitutive Equation

$$p = p_0 + \rho' \left[\frac{\partial p}{\partial \rho}\right]_S + \frac{1}{2} (\rho')^2 \left[\frac{\partial^2 p}{\partial \rho^2}\right]_S + \cdots$$

Speed of Sound

$$c^2 \equiv \left[\frac{\partial p}{\partial \rho}\right]_S \quad \text{(Sound speed)}$$

The Linear Wave Equation

$$\begin{aligned} \frac{\partial \rho'}{\partial t} &= -\rho_0 \nabla \cdot \mathbf{v} \,, \\ \frac{\partial \mathbf{v}}{\partial t} &= -\frac{1}{\rho_0} \nabla p'(\rho) \,, \\ p' &= \rho' c^2 \,. \end{aligned}$$

Wave Equation for Pressure

$$\rho \nabla \cdot \left(\frac{1}{\rho} \nabla p\right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \,,$$

The Acoustic Wave Equation

Wave Equation for Pressure

$$\rho \, \nabla \cdot \left(\frac{1}{\rho} \nabla p \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \, ,$$

Wave Equation for Particle Velocity

$$\frac{1}{\rho} \nabla \left(\rho c^2 \, \nabla \cdot \mathbf{v} \right) - \frac{\partial^2 \mathbf{v}}{\partial t^2} = \mathbf{0} \, .$$

Wavefield Potentials

Wave Equation for Velocity Potential

Constant density ρ:

$$\mathbf{v} = \nabla \phi \, .$$

$$\nabla \left(c^2 \, \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} \right) = \mathbf{0} \, .$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \, ,$$

Wave Equation for Displacement Potential

$$\mathbf{u} = \nabla \psi \, ,$$

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \, .$$

$$p = -K \, \nabla^2 \psi \, ,$$

$$K = \rho c^2 \, .$$

$$p = -\rho \frac{\partial^2 \psi}{\partial t^2} \, .$$

The Acoustic Wave Equation

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \, ,$$

- 4-D Partial Differential Equation
- Analytical solutions only for few canonical problems
- Direct Numerical Solution (FDM, FEM)
  - Computationally intensive ($\Delta x \ll \lambda \, , \; \Delta t \ll T$).
- Dimension Reduction for PDE
  - Geometrical symmetries (Plane or axisymmetric problems)
  - Integral transforms
  - Analytical or numerical solution of ODE or low dimensional PDE.
  - Evaluation of inverse transforms (analytical or numerical)

The Helmholtz Equation

Frequency–time Fourier transform pair

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\, e^{-i\omega t} d\omega \,,$$

$$f(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt \,,$$

Helmholtz Equation

$$\left[\nabla^2 + k^2(\mathbf{r})\right] \psi(\mathbf{r},\omega) = 0 \,,$$

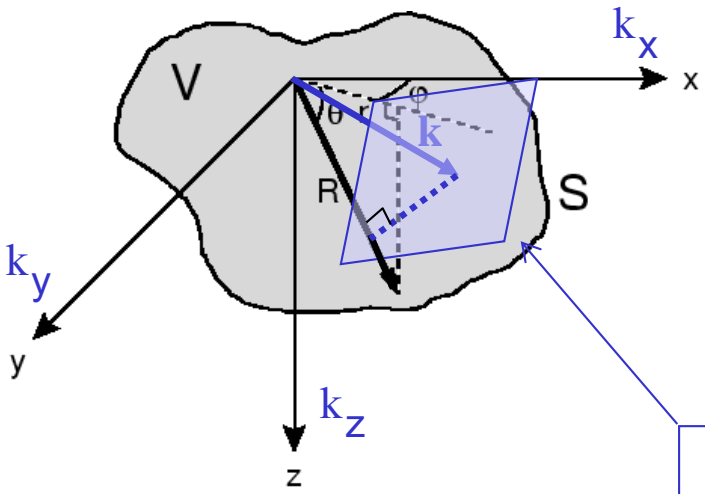
Medium wavenumber

$k(\mathbf{r}) = \omega / c(\mathbf{r})$

Solution of Helmholtz Equation depends on

- Dimensionality of propagation problem
- Medium wavenumber/sound speed variation
- Boundary geometry and conditions
- Source-receiver geometry
- Frequency and bandwidth

The Helmholtz Equation



k: Wave vector
Propagation direction

Helmholtz Equation for Homogeneous Media

Cartesian Coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} ,$$

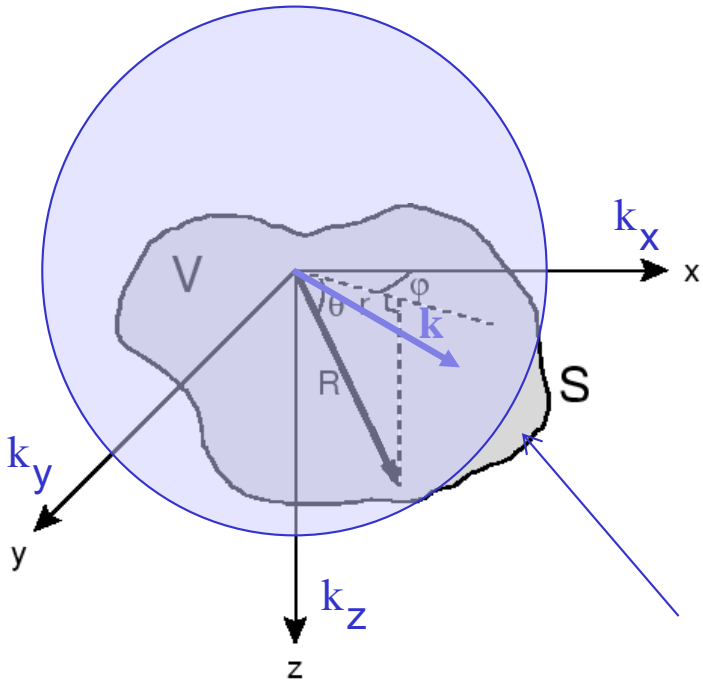
$$\psi(x, y, z) = \begin{cases} A\, e^{i\mathbf{k}\cdot\mathbf{r}} \\ B\, e^{-i\mathbf{k}\cdot\mathbf{r}} \end{cases} ,$$

Wavefronts: $\mathbf{k} \cdot \mathbf{r} = \text{const}$

1-D propagation: $k_y, k_z = 0$:

$$\psi(x) = \begin{cases} A\, e^{ikx} & \text{Forward propagating} \\ B\, e^{-ikx} & \text{Backward propagating} \end{cases}$$

The Helmholtz Equation



Spherical Cordinates

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+k^2\right]\psi(r)=0\,,$$

$$\psi(r)=\begin{cases}(A/r)\,e^{ikr} & \text{Diverging waves}\\ (B/r)\,e^{-ikr} & \text{Converging waves}\end{cases}$$

Cylindrical Cordinates

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\,.$$

Axial Symmetry

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+k^2\right]\psi(r)=0\,,$$

bessel Functions

$$\psi(r)=\begin{cases}A\,J_0(kr)\\ B\,Y_0(kr)\,,\end{cases}$$

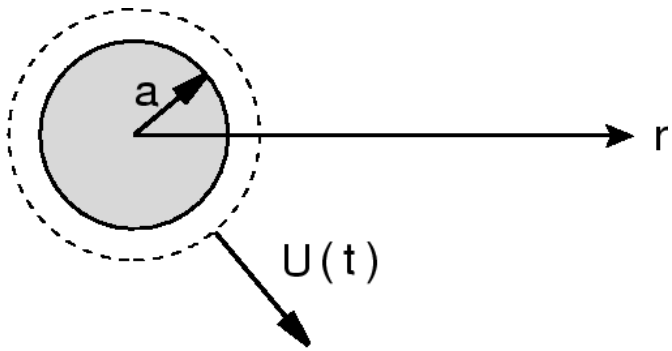
Hankel Functions

$$\psi(r)=\begin{cases}CH_0^{(1)}(kr) &= C\left[J_0(kr)+iY_0(kr)\right]\\ DH_0^{(2)}(kr) &= D\left[J_0(kr)-iY_0(kr)\right]\,.\end{cases}$$

$$H_0^{(1)}(kr)\simeq\sqrt{\frac{2}{\pi kr}}\,e^{i(kr-\pi/4)}\ \text{Diverging waves}$$

$$H_0^{(2)}(kr)\simeq\sqrt{\frac{2}{\pi kr}}\,e^{-i(kr-\pi/4)}\ \text{Converging waves}$$

Source in Unbounded Medium



Frequency Domain

$$u_r(a) = U(\omega)\,.$$

Spherical geometry solution

$$\begin{aligned}
\psi(r) &= A\,\frac{e^{ikr}}{r}\,, \\
u_r(r) &= \frac{\partial\psi(r)}{\partial r} = A\,e^{ikr}\left(\frac{ik}{r} - \frac{1}{r^2}\right).
\end{aligned}$$

Simple point source: $ka \ll 1$

$$\begin{aligned}
u_r(a) &= A\,e^{ika}\,\frac{ika-1}{a^2} \simeq -\frac{A}{a^2}\,, \\
A &= -a^2\,U(\omega)\,. \\
&\Rightarrow \\
\psi(r) &= -S_\omega\,\frac{e^{ikr}}{4\pi r}\,. \\
S_\omega &= 4\pi a^2\,U(\omega)
\end{aligned}$$