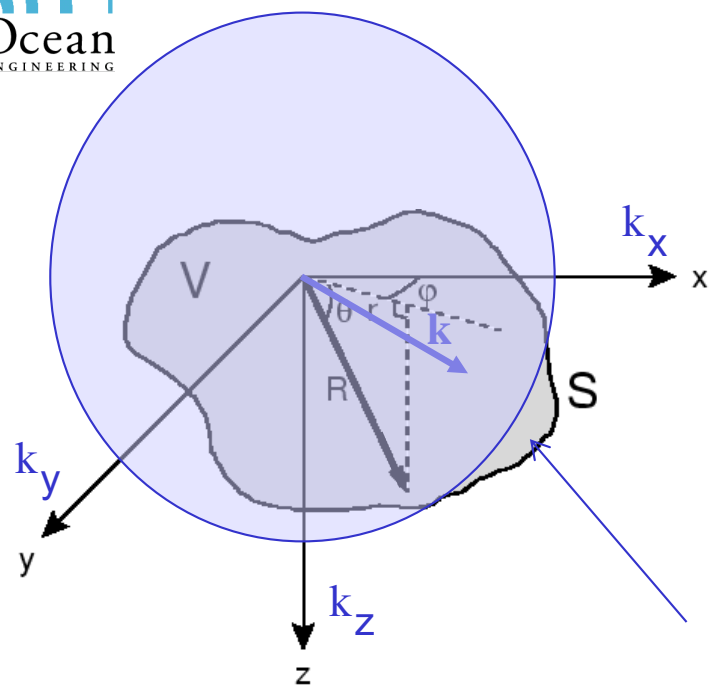
The Helmholtz Equation



Spherical Cordinates

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+k^2\right]\psi(r)=0\,,$$

$$\psi(r)=\begin{cases}(A/r)\,e^{ikr} & \text{Diverging waves}\\ (B/r)\,e^{-ikr} & \text{Converging waves}\end{cases}$$

Cylindrical Cordinates

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\,.$$

Axial Symmetry

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+k^2\right]\psi(r)=0\,,$$

bessel Functions

$$\psi(r)=\begin{cases}A\,J_0(kr)\\ B\,Y_0(kr)\,,\end{cases}$$

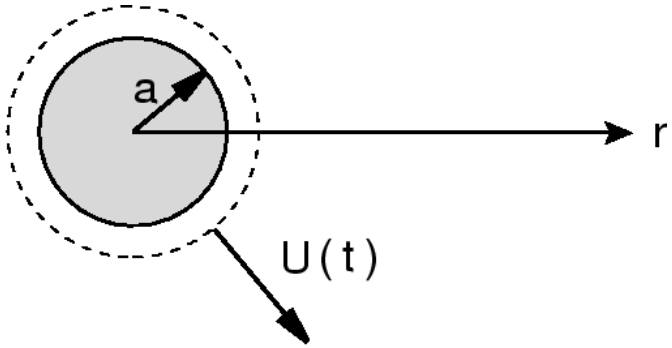
Hankel Functions

$$\psi(r)=\begin{cases}CH_0^{(1)}(kr) &= C\left[J_0(kr)+iY_0(kr)\right]\\ DH_0^{(2)}(kr) &= D\left[J_0(kr)-iY_0(kr)\right]\,.\end{cases}$$

$$H_0^{(1)}(kr)\simeq\sqrt{\frac{2}{\pi kr}}\,e^{i(kr-\pi/4)}\ \text{Diverging waves}$$

$$H_0^{(2)}(kr)\simeq\sqrt{\frac{2}{\pi kr}}\,e^{-i(kr-\pi/4)}\ \text{Converging waves}$$

Point Source in Unbounded Medium



Green's function

$$g_\omega(r,0) = \frac{e^{ikr}}{4\pi r}\,,$$

Source at r_0

$$g_\omega(\mathbf{r},\mathbf{r}_0) = \frac{e^{ikR}}{4\pi R}\,, \quad R = |\mathbf{r}-\mathbf{r}_0|\,.$$

Helmholtz Equation for Green's function

$$\left[\nabla^2 + k^2\right] g_\omega(\mathbf{r},\mathbf{r}_0) = -\delta(\mathbf{r}-\mathbf{r}_0)\,,$$

Integrate over spherical volume V of radius $\epsilon \to 0$:

$$\begin{aligned}
\int_V -\delta(\mathbf{r}-\mathbf{r}_0)dV &= -1 \\
\int_V k^2 g_\omega(\mathbf{r},\mathbf{r}_0)dV &\to_{\epsilon\to 0} 0 \\
\int_V \nabla^2 g_\omega(\mathbf{r},\mathbf{r}_0)dV &= \int_S \frac{\partial}{\partial R} g_\omega(\mathbf{r},\mathbf{r}_0)dS \\
&= \int_S \left[-\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon}\right] dS \\
&= 4\pi\epsilon^2\left[-\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon}\right] \to_{\epsilon\to 0} -1
\end{aligned}$$

Reciprocity

$$g_\omega(\mathbf{r},\mathbf{r}_0) = g_\omega(\mathbf{r}_0,\mathbf{r})\,,$$

Ocean Acoustic Theory

	JKPS
• Acoustic Wave Equation	
• Integral Transforms	
• Helmholtz Equation	
• Source in Unbounded Media	2.3.1-2.3.2
• Source in Bounded Media	2.3.3
• Propagation in Layered Media	2.4
– Reflection and Transmission	2.4.1-2.4.3
– The Ideal Waveguide	2.4.4
• Image Method	
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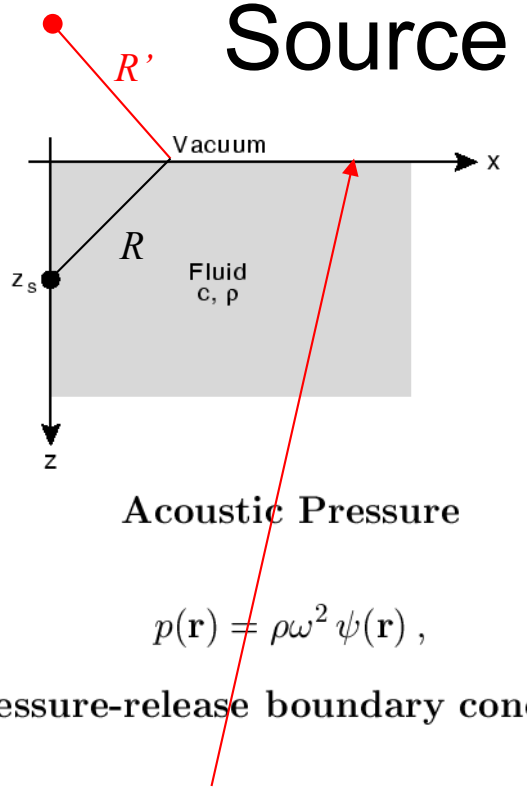
• Normal Modes	
– Pekeris Waveguide	2.4.5

Green's Theorem
Source in Fluid Halfspace



Acoustic Pressure

$$p(\mathbf{r}) = \rho\omega^2\,\psi(\mathbf{r})\,,$$

Pressure-release boundary condition

$$\psi(\mathbf{r}_0) \equiv 0\,, \quad \mathbf{r}_0 = (x, y, 0)\,.$$

Green's theorem

$$\psi(\mathbf{r}) = \int_S G_\omega(\mathbf{r}, \mathbf{r}_0)\,\frac{\partial\psi(\mathbf{r}_0)}{\partial\mathbf{n}_0}\,dS_0 - \int_V f(\mathbf{r}_0)\,G_\omega(\mathbf{r}, \mathbf{r}_0)\,dV_0\,.$$

Simple point source

Eq (2.63)

$$f(\mathbf{r}_0) = S_\omega\,\delta(\mathbf{r}_0 - \mathbf{r}_s)\,.$$

Green's Function

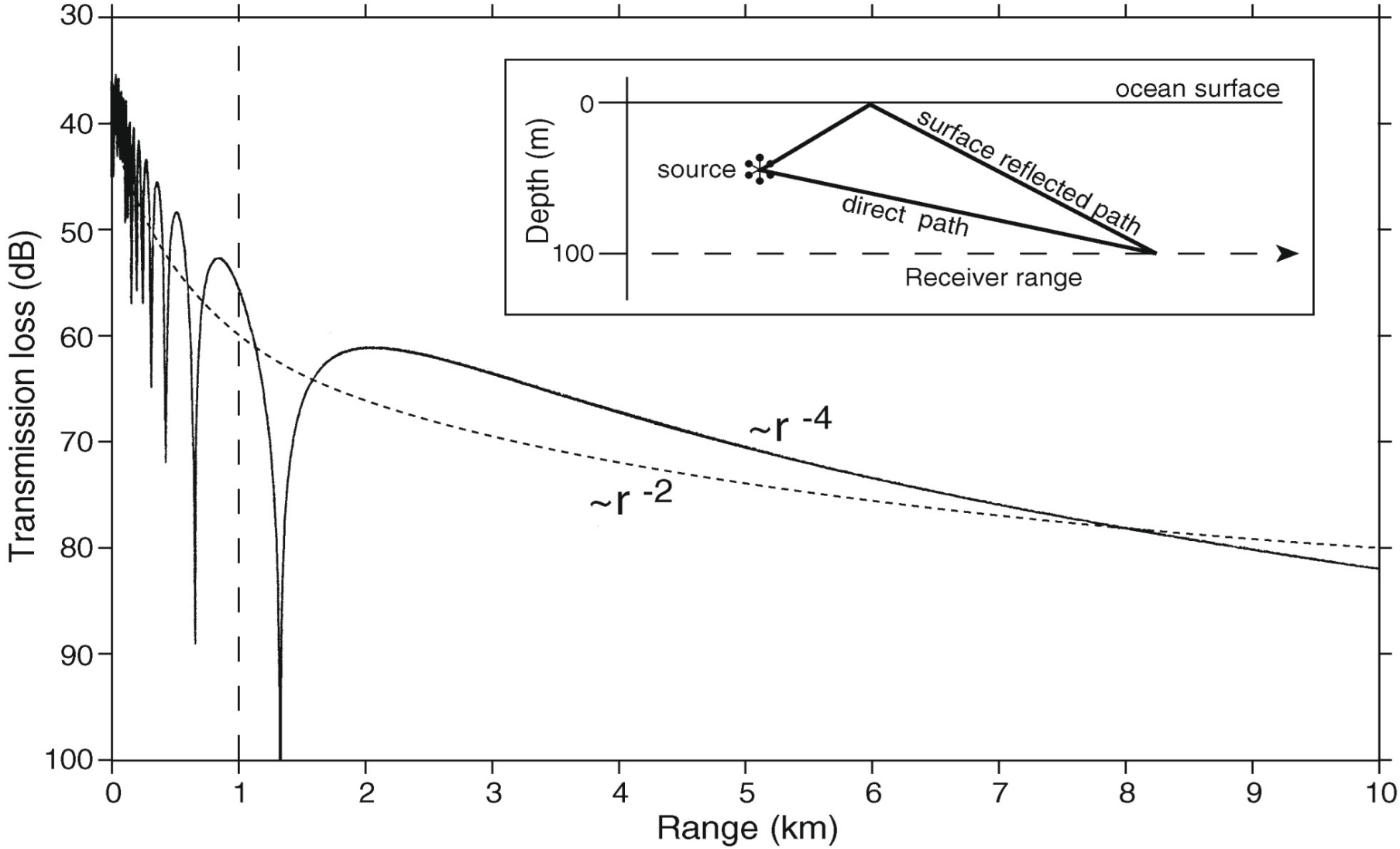
$$\begin{aligned}
\text{Choose}\quad G_\omega(\mathbf{r}, \mathbf{r}_0) &\equiv 0 \quad \text{for} \quad \mathbf{r}_0 = (x, y, 0) \\
G_\omega(\mathbf{r}, \mathbf{r}_0) &= g_\omega(\mathbf{r}, \mathbf{r}_0) + H_\omega(\mathbf{r}) \\
&= \frac{e^{ikR}}{4\pi R} - \frac{e^{ikR'}}{4\pi R'} \\
&\Rightarrow \\
\psi(\mathbf{r}) &= -S_\omega\,G_\omega(\mathbf{r}, \mathbf{r}_s)\,.
\end{aligned}$$

$$\begin{aligned}
R &= \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}\,, \\
R' &= \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z + z_s)^2}\,.
\end{aligned}$$

Acoustic Pressure

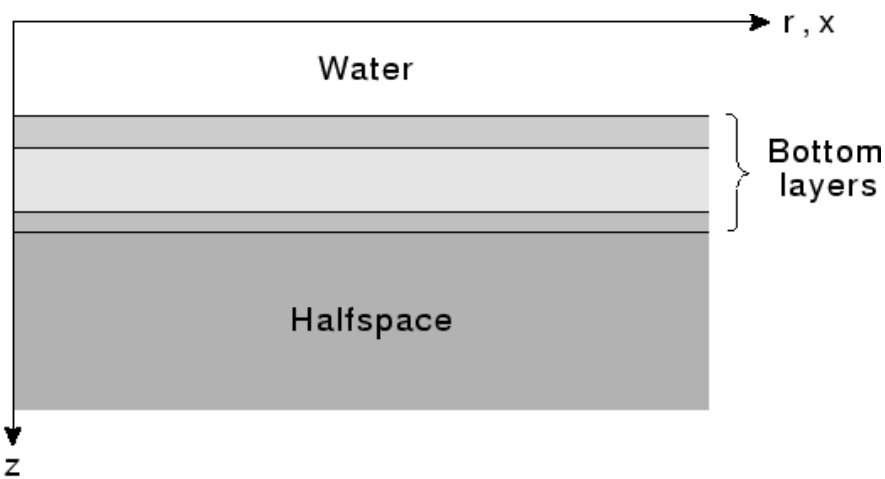
$$p(\mathbf{r}) = \rho\omega^2\psi(\mathbf{r}) = -\rho\omega^2 S_\omega\left[\frac{e^{ikR}}{4\pi R} - \frac{e^{ikR'}}{4\pi R'}\right]\,,$$

LLOYD MIRROR EFFECT



Layered Media and Waveguides

Range-independent Environments



Integral Transform Solution

Helmholtz Equation - Layer n

$$\left[\nabla^2 + k_n^2(z)\right]\psi(\mathbf{r}) = f(\mathbf{r})\,,$$

Interface Boundary Conditions

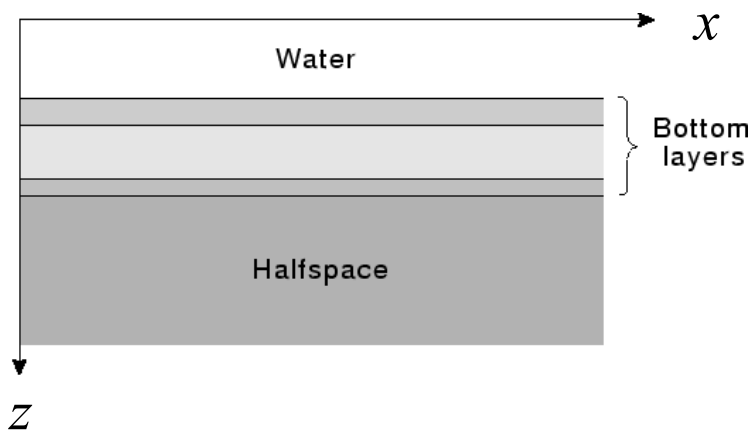
$$B\left[\psi(\mathbf{r})\right]\big|_{z=z_n} = 0\,, \quad n = 1 \cdots N\,,$$

Plane geometry: Fourier Transform

$$\left[\nabla^2 + k_n^2(z)\right]\psi(\mathbf{r}) = f(\mathbf{r})$$

$$f(x,z) = \int_{-\infty}^{\infty} f(k_x,z)\, e^{ik_x x}\, dk_x \,,$$

$$f(k_x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x,z)\, e^{-ik_x x}\, dx \,,$$



Depth-Separated Wave Equation

$$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right]\psi(k_x,z) = S_\omega \frac{\delta(z-z_s)}{2\pi} \,.$$

Depth-Separated Green's Function

$$\psi(k_x,z) = -S_\omega\, G_\omega(k_x,z,z_s) = -S_\omega\left[g_\omega(k_x,z,z_s) + H_\omega(k_x,z)\right]$$

Superposition Principle

Source in free space
+
Homogeneous solution accounting for boundary interactions

$$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] g_\omega(k_x,z,z_s) = -\frac{\delta(z-z_s)}{2\pi}$$

$$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] H_\omega(k_x,z) = 0$$

Interface Boundary Conditions

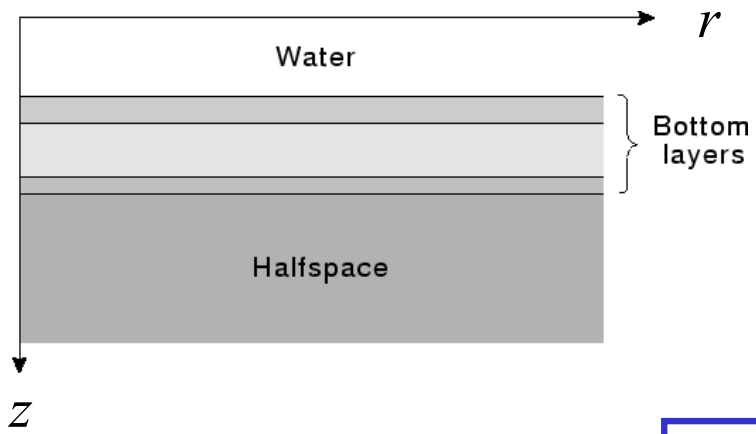
$$B\left[\psi(k_x,z_n)\right] = 0 \,.$$

Axisymmetry: Hankel Transform

$$\left[\nabla^2 + k_n^2(z)\right]\psi(\mathbf{r}) = f(\mathbf{r})$$

$$f(r,z) = \int_0^\infty f(k_r,z)\, J_0(k_r r)\, k_r\, dk_r\,,$$

$$f(k_r,z) = \int_0^\infty f(r,z)\, J_0(k_r r)\, r\, dr\,,$$



Depth-Separated Wave Equation

$$\left[\frac{d^2}{dz^2} + (k^2 - k_r^2)\right]\psi(k_r,z) = S_\omega\,\frac{\delta(z-z_s)}{2\pi}\,.$$

Depth-Separated Green's Function

$$\psi(k_r,z) = -S_\omega\, G_\omega(k_r,z,z_s) = -S_\omega\left[g_\omega(k_r,z,z_s) + H_\omega(k_r,z)\right]$$

Superposition Principle

Source in free space
+
Homogeneous solution accounting for boundary interactions

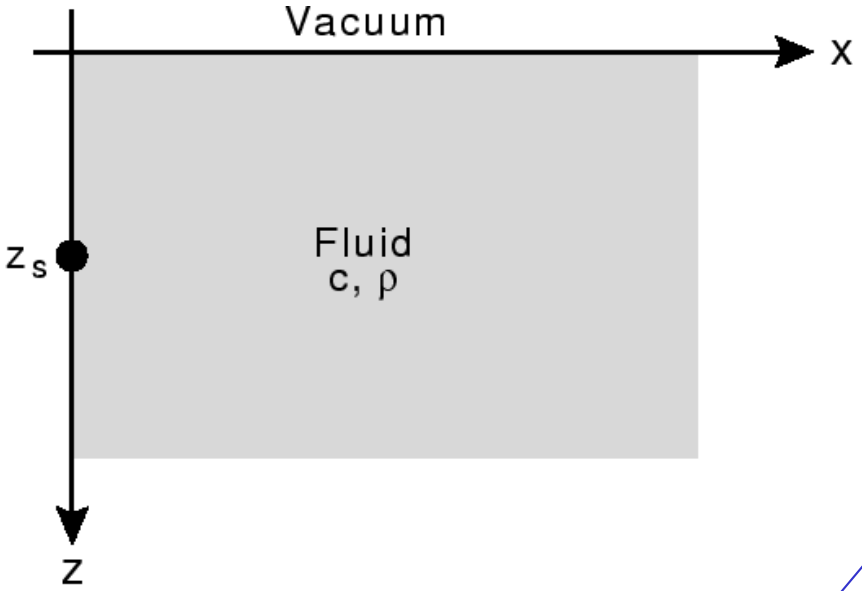
$$\left[\frac{d^2}{dz^2} + (k^2 - k_r^2)\right] g_\omega(k_r,z,z_s) = -\frac{\delta(z-z_s)}{2\pi}$$

$$\left[\frac{d^2}{dz^2} + (k^2 - k_r^2)\right] H_\omega(k_r,z) = 0$$

Interface Boundary Conditions

$$B\left[\psi(k_r,z_n)\right] = 0\,.$$

Example
Source in Fluid Halfspace



$$\psi(k_r, z) = -S_\omega\, G_\omega(k_r, z, z_s) = -S_\omega\,[g_\omega(k_r, z, z_s) + H_\omega(k_r, z)]$$

$$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] g_\omega(k_x, z, z_s) = -\frac{\delta(z - z_s)}{2\pi}$$

$$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] H_\omega(k_x, z) = 0$$

Homogeneous Solution

$$H_\omega(k_r, z) = A^+(k_r)\, e^{ik_z z} + A^-(k_r)\, e^{-ik_z z}\,,$$

Vertical Wavenumber

$$\begin{aligned} k_z &= \sqrt{k^2 - k_r^2} \\ &= \begin{cases} \sqrt{k^2 - k_r^2}\,, & k_r \le k \\ i\sqrt{k_r^2 - k^2}\,, & k_r > k\,. \end{cases} \end{aligned}$$

Radiation Conditions

$$H_\omega(k_r, z) = \begin{cases} A^+(k_r)\, e^{ik_z z}\,, & z \to +\infty \\ A^-(k_r)\, e^{-ik_z z}\,, & z \to -\infty\,. \end{cases}$$

Source in Infinite Medium

Source field

$$g_\omega(k_r, z, z_s) = A(k_r) \begin{cases} e^{ik_z(z-z_s)}, & z \geq z_s \\ e^{-ik_z(z-z_s)}, & z \leq z_s \end{cases}$$

$$= A(k_r)\, e^{ik_z|z-z_s|} .$$

Integration of depth-separated wave equation over $[z_s - \epsilon, z_s + \epsilon]$:

$$\left[\frac{dg_\omega(k_r, z)}{dz}\right]_{z_s-\epsilon}^{z_s+\epsilon} + O(\epsilon) = -\frac{1}{2\pi} .$$

$$\Rightarrow$$

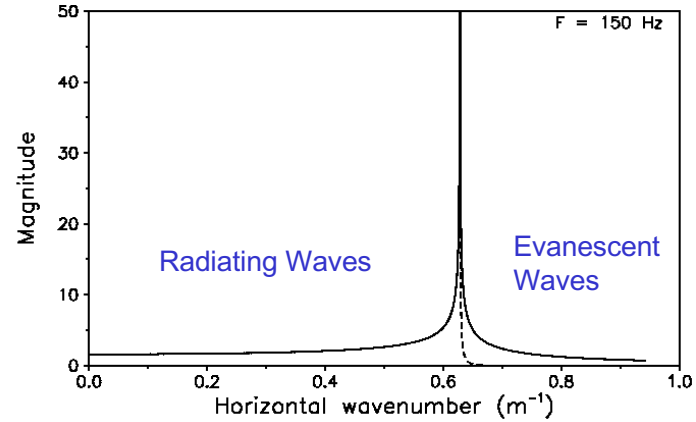
$$A(k_r) = -\frac{1}{4\pi i k_z}$$

$$\Rightarrow$$

$$g_\omega(k_r, z, z_s) = -\frac{e^{ik_z|z-z_s|}}{4\pi i k_z} .$$

Inverse Hankel Transform - Sommerfeld–Weyl Integral

$$g_\omega(r, z, z_s) = \frac{i}{4\pi} \int_0^\infty \frac{e^{ik_z|z-z_s|}}{k_z} J_0(k_r r)\, k_r\, dk_r ,$$



Grazing Angle Representation

$$k_x = k\cos\theta ,$$

$$k_z = k\sin\theta ,$$

$$\frac{dk_x}{d\theta} = -k_z .$$

$$\Rightarrow$$

$$g_\omega(\mathbf{r}, \mathbf{r}') \simeq \frac{i}{4\pi} \int_{-k}^{k} \frac{e^{ik_z|z-z_s|}}{k_z} e^{ik_x x}\, dk_x$$

$$= \frac{i}{4\pi} \int_0^\pi e^{ik|z-z_s|\sin\theta + ikx\cos\theta}\, d\theta .$$

Source in Fluid Halfspace

Halfspace Problem: Surface and Radiation Conditions

$$\psi(k_r, 0) \equiv 0$$

$$\psi(k_r, z) \quad \text{radiating for} \quad z \to \infty$$

$$\Rightarrow$$

$$\begin{aligned} \psi(k_r, 0) &= -S_\omega \left[g_\omega(k_r, 0, z_s) + H_\omega(k_r, 0) \right] \\ &= S_\omega \left[\frac{e^{ik_z z_s}}{4\pi i k_z} - A^+(k_r) \right] = 0 \,, \end{aligned}$$

Total field

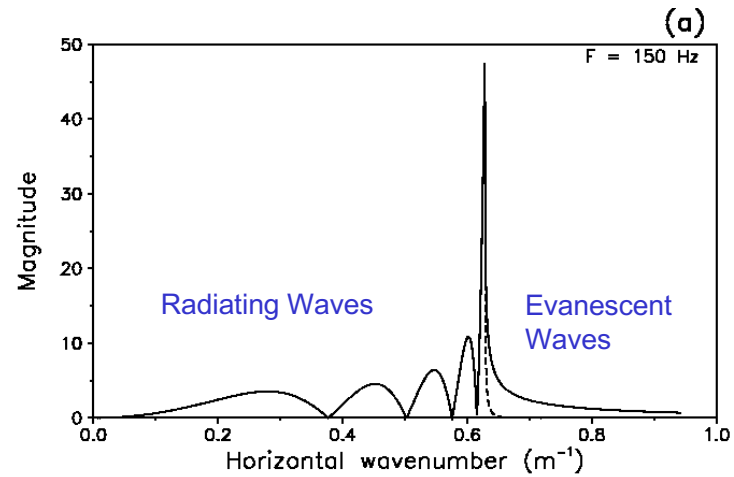
$$\psi(k_r, z) = S_\omega \left[\frac{e^{ik_z |z - z_s|}}{4\pi i k_z} - \frac{e^{ik_z (z + z_s)}}{4\pi i k_z} \right] .$$

Loyd-Mirror Minima and Maxima

$$\begin{aligned} \sin\theta_{\max} &= \frac{(2m-1)\,\pi}{2kz_s} \,, \\ \sin\theta_{\min} &= \frac{(m-1)\,\pi}{kz_s} \,. \end{aligned}$$

Free Surface Reflection Coefficient

$$R = -1 \,.$$



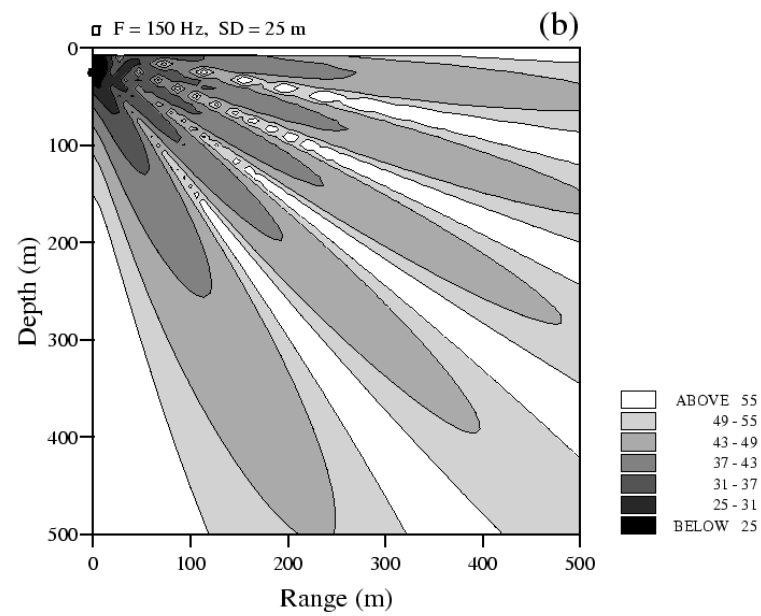


Fig. 2.7. Source in a fluid halfspace. **(a)** Magnitude of the depth-dependent Green's function. Solid curve: $z - z_s = \lambda/10$. Dashed curve: $z - z_s = 2\lambda$. **(b)** Pressure field contours.