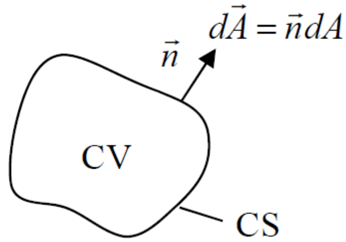
Lesson 1: Application of Reynolds Transport Theorem (Flow in a pipe)

Reynolds theorem for an arbitrary (scalar or vector) variable β.

$$\frac{d}{dt}\int_{S(t)} \beta\rho dV = \frac{\partial}{\partial t}\int_{CV} \beta\rho dV + \int_{CS} \beta\rho\vec{v}d\vec{A} \quad (1)$$



a) <u>Continuity</u>

Take $B = m, \beta = \frac{dB}{dm} = 1$ in (1)

$$\int_{CV} \beta\rho dV = \int_{CV} \rho dV = m = \text{mass}$$

Physical Law: Conservation of mass

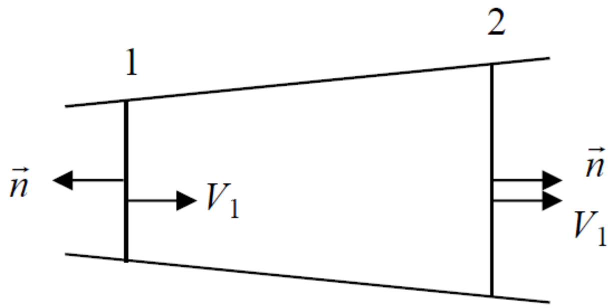
$$\frac{dm}{dt} = 0$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho dV + \int_{CS} \rho\vec{v}d\vec{A}$$

V=volume bounded by *C.S.*

$$0 = \rho\frac{\partial V}{\partial t} + \int_{CS} \rho\vec{v}d\vec{A}$$

For a pipe: on the lateral walls $\vec{v} \perp \vec{A}$, so $\vec{v}d\vec{A} = 0$



$$\int_{CS} \rho \vec{v} d\vec{A} = \int_1 \rho \vec{v} d\vec{A} + \int_2 \rho \vec{v} d\vec{A} = -\rho V_1 A_1 + \rho V_2 A_2$$

$$\Rightarrow -\frac{\partial V}{\partial t} = -V_1 A_1 + V_2 A_2 = -Q_1(t) + Q_2(t)$$

For the case $V(t) = \text{const.}$ (rigid wall)

$$Q_2(t) - Q_1(t) = 0 \Rightarrow Q_1 = Q_2$$

b) Momentum

Physical Law: Newton's second law of motion

$$\sum \vec{F} = m\vec{a} = \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}\int_{CV} \rho \vec{v} dV$$

Take $B = m\vec{v}$, $\beta = \frac{dB}{dm} = \vec{v}$ in (1)

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_{CV} \rho \vec{v} dV + \int_{CS} \rho \vec{v}\left(\vec{v} d\vec{A}\right) \qquad (2)$$

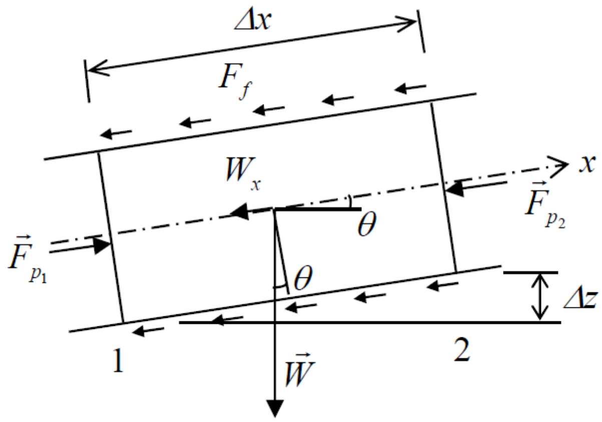
If flow is steady, i.e. $\frac{\partial}{\partial t} = 0$

Apply (2) along *x*-axis (streamwise direction in a pipe)

$$\sum F_x = \int_{CS} \rho v_x (\vec{v} d\vec{A}) \quad (3)$$

Application: shear stress distribution in a circular pipe of uniform cross-section inclined at an angle θ with horizontal

Consider steady flow in a pipe of radius R



Consider cylindrical volume element of radius $r < R$ and length Δx.

Assuming streamlines are parallel to the pipe centerline for any cylindrical CV, $A1 = A2 = A$, $V1 = V2 = Q/A$. From (3)

$$F_{p1} - F_{p2} - W_x - F_f = \rho V_1(-V_1 A_1) + \rho V_2(V_2 A_2) = 0 \quad (4)$$

Use

$$p_1 = p(x)$$

$$p_2 = p(x + \Delta x) = p(x) + \frac{dp}{dx}\Delta x = p_1 + \frac{dp}{dx}\Delta x$$

$$\sin\theta = \frac{dz}{dx}$$

$$F_{p1} - F_{p2} = p_1 A_1 - p_2 A_2 = -\frac{dp}{dx} A\Delta x$$

$$F_f = \tau(r) \cdot (lateral\ surface\ of\ CV) = \tau(2\pi r \Delta x)$$

$$W_x = W sin\theta = \gamma V sin\theta = \gamma A \Delta x sin\theta$$

Plug in (4), divide by $(\Delta x \cdot \pi r)$

$$\tau(r) = \frac{\gamma r}{2}\left[-\frac{d}{dx}\left(\frac{p}{r} + z\right)\right]$$

For $r = 0, \tau(r = 0) = 0$.

For $r = 0, \tau(r = R) = \tau_0 = \frac{\gamma R}{2}\left[-\frac{d}{dx}\left(\frac{p}{r} + z\right)\right]$.

Also for a circular pipe as $V_1 = V_2$

$dH = d\left(\frac{p}{r} + z + \frac{V^2}{2g}\right) = d\left(\frac{p}{r} + z\right), H =$ total energy (head)

$$\tau_0 = \frac{\gamma R}{2}\left[-\frac{dH}{dx}\right] \Rightarrow \frac{2\tau_0}{\gamma R} = -\frac{dH}{dx} = -\frac{\Delta H_{12}}{\Delta x_{12}} = \frac{h_{f12}}{\Delta x_{12}}$$

$h_f =$ head loss due to friction forces as difference in total head (energy) between two sections. i.e. the head (energy) lost between the same two sections

$$h_f = \frac{2\tau_0}{\gamma R}\Delta x = \frac{4\tau_0}{\gamma D}\Delta x \Rightarrow h_f \sim \tau_0$$

c) <u>Energy</u>

Energy equation for steady incompressible flow between two sections

$$z_1 + \frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + h_p = z_2 + \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + h_t + h_{f12}$$

z = elevation above datum of section centroid

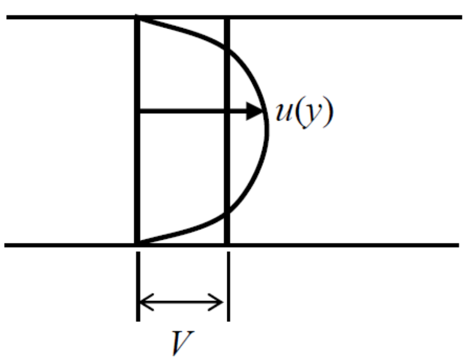
α = kinetic energy correction factor = $\frac{\int_A u^3 dA}{V^3 A}$, where V is the averaged velocity and $V = \frac{\int_A udA}{VA}$.

h_p = head added by pumps

h_t = head removed by turbine

h_{f12} = head loss due to friction

If flow in the section is uniform α = 1.



For laminar (parabolic) profile α = 2.

For turbulent (logarithmic or power low) profile α = 1.03-1.06.

In many applications of turbulent flow in pipe, assume α = 1.

For turbulent flow in a rectangular open channel α = 1.5-2.