海岸动力地貌学

第四章 海岸动力过程

波浪边界层及底床剪应力介绍

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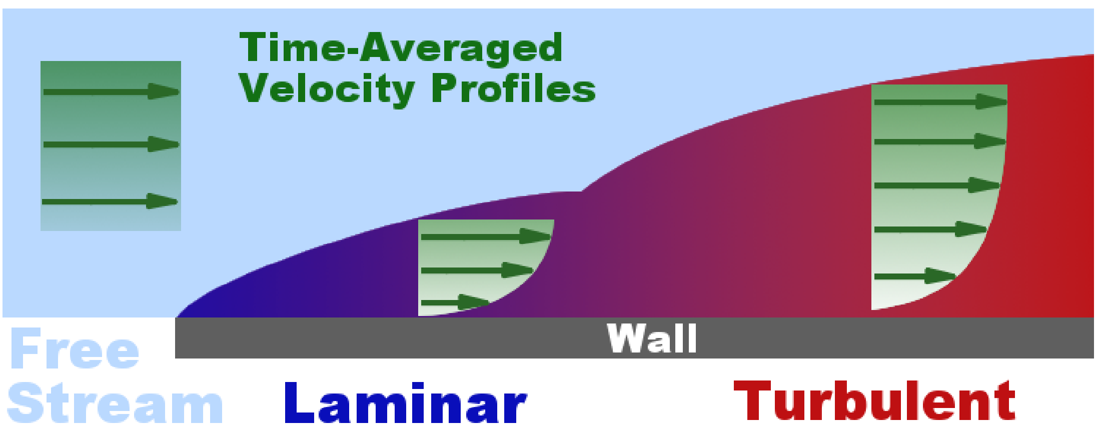
导读

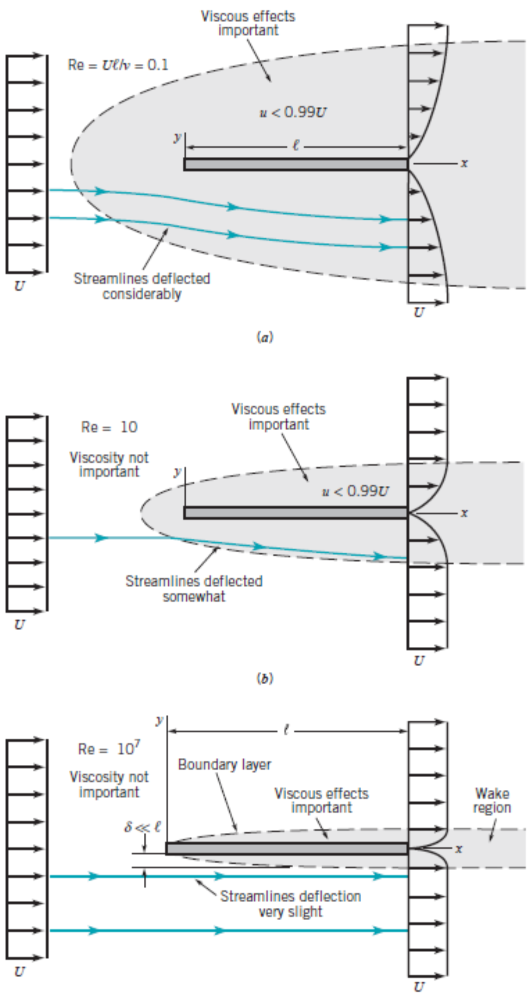
- **海浪**是塑造海岸地貌过程中最积极、最活跃的动力因素。它产生的巨大能量对海岸或海岸建筑物产生很强的冲击力，在**近岸物质搬运和堆积**方面也有重要作用。**潮汐**引起的海水周期性升降运动以及随之产生的**海水水平方向运动**，对塑造海岸地貌也有重要影响，其他的各种动力在不同时空尺度上对海岸也有著或大或小的影响。

Review of boundary layer theory

What is a ***boundary layer***?

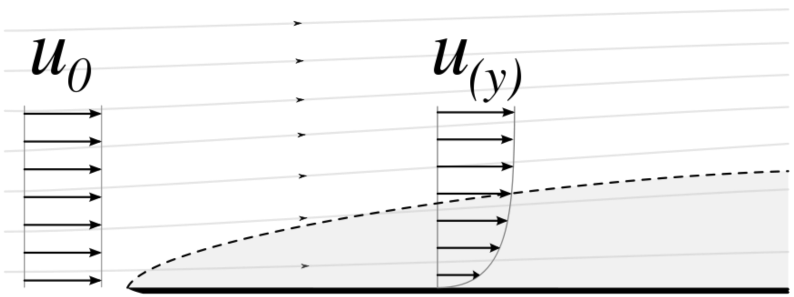
A boundary layer is the layer of fluid in the immediate vicinity of a bounding surface where the effects of viscosity are significant.





Prandtl/Blasius Boundary Layer Solution

For steady, two-dimensional laminar flows with negligible gravitational effects:



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

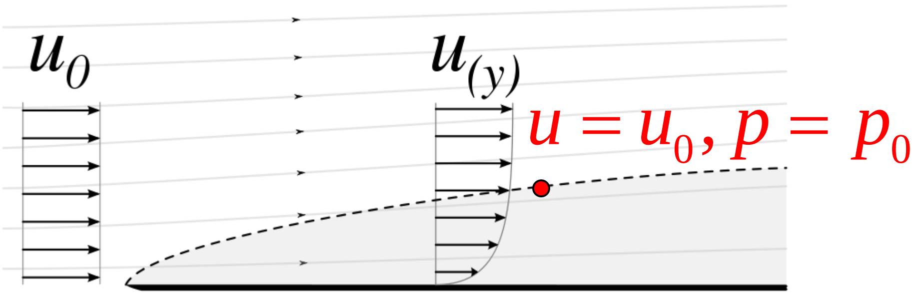
Within the boundary layer:

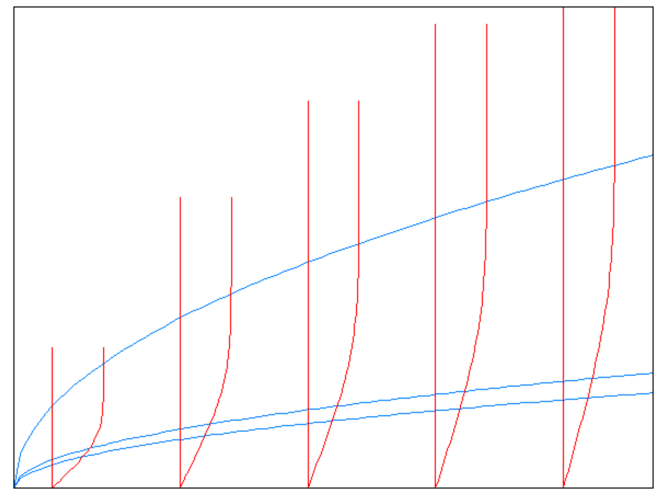
$$v \ll u, \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y} \Rightarrow \frac{\partial v}{\partial x} \approx 0, \frac{\partial v}{\partial y} \approx 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

Prandtl/Blasius Boundary Layer Solution





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At the edge of B.L.:

$$u_0 \cancel{\frac{\partial u_0}{\partial x}} + v_0 \cancel{\frac{\partial u_0}{\partial y}} = -\frac{1}{\rho}\frac{\partial p_0}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u_0}{\partial x^2}} + \cancel{\frac{\partial^2 u_0}{\partial y^2}}\right)$$

$$\frac{\partial p_0}{\partial x} = \frac{\partial p}{\partial x} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

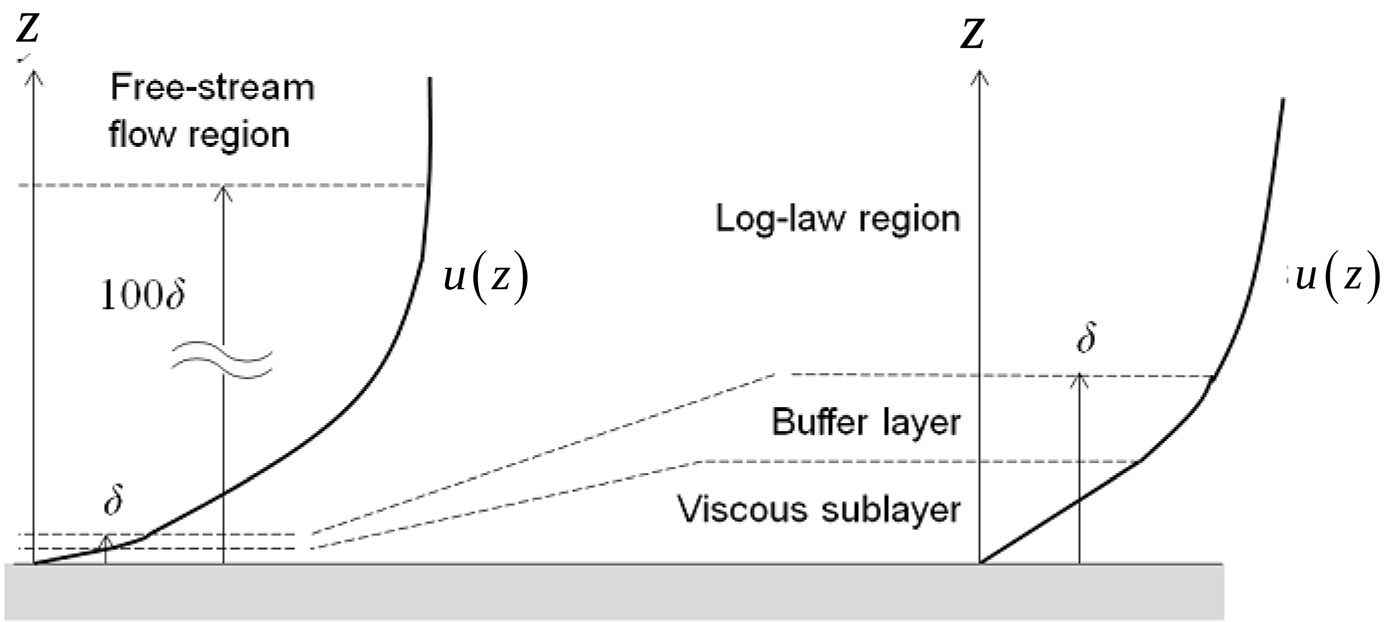
$$\eta = \frac{y}{\delta(x)} = y\sqrt{\frac{U}{2\nu x}} \qquad \psi = \sqrt{2\nu U x}\, f(\eta)$$

$$u(x,y) = \frac{\partial \psi}{\partial y} = U f'(\eta)$$

$$v(x,y) = -\frac{\partial \psi}{\partial x} = \sqrt{\frac{\nu U}{2x}}(\eta f' - f)$$

$$f''' + f'' f = 0$$

Law of the wall (Log-law region)

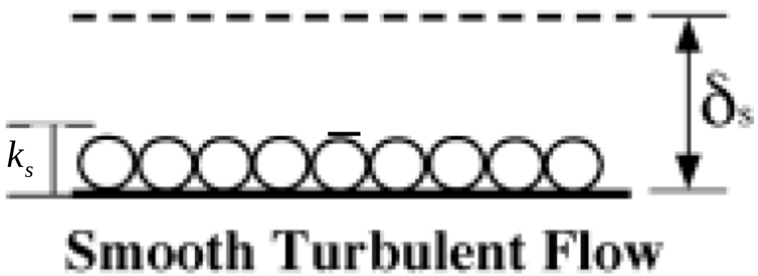


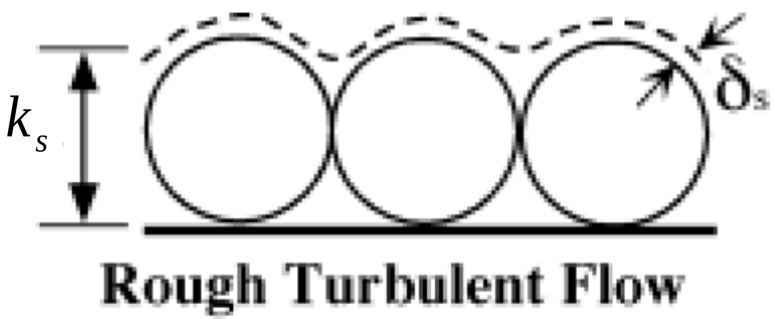
Log-law region: $u = \frac{u_*}{\kappa}\ln\left(\frac{z}{z_0}\right), \tau = \rho u_*^2$ $\tau = \mu \left.\frac{\partial u}{\partial z}\right|_{z=0}$

u_*: friction velocity; z_0: the distance from the boundary to the location of zero velocity; τ: bottom (wall) shear stress; κ:the Von Kármán constant (=0.40~0.41)

Law of the wall (Log-law region)

Log-law region: $u = \frac{u_*}{\kappa}\ln\left(\frac{z}{z_0}\right)$





$$\frac{u_* k_s}{\nu} < 5, z_0 = \frac{\nu}{9u_*}$$

$$\frac{u}{u_*} = 5.75\log\left(\frac{u_* z}{\nu}\right) + 5.5$$

$$\frac{u_* k_s}{\nu} > 70 - 100, z_0 = \frac{k_s}{30}$$

$$\frac{u}{u_*} = 5.75\log\left(\frac{30z}{k_s}\right)$$

$$k_s \cong 3d_{90}\,(\text{Julie,1998})$$

A) Small-amplitude (linear, Airy) wave theory (Airy, 1844)

$\varsigma = a\cos\theta$: surface elevation/displacement

$a = H/2$: wave amplitude; $\theta = kx - \sigma t$: phase angle

$k = 2\pi/L$: wave number; $\sigma = 2\pi/T = 2\pi f$: angular frequency

$\sigma^2 = gk\tanh(kh)$: dispersion relation

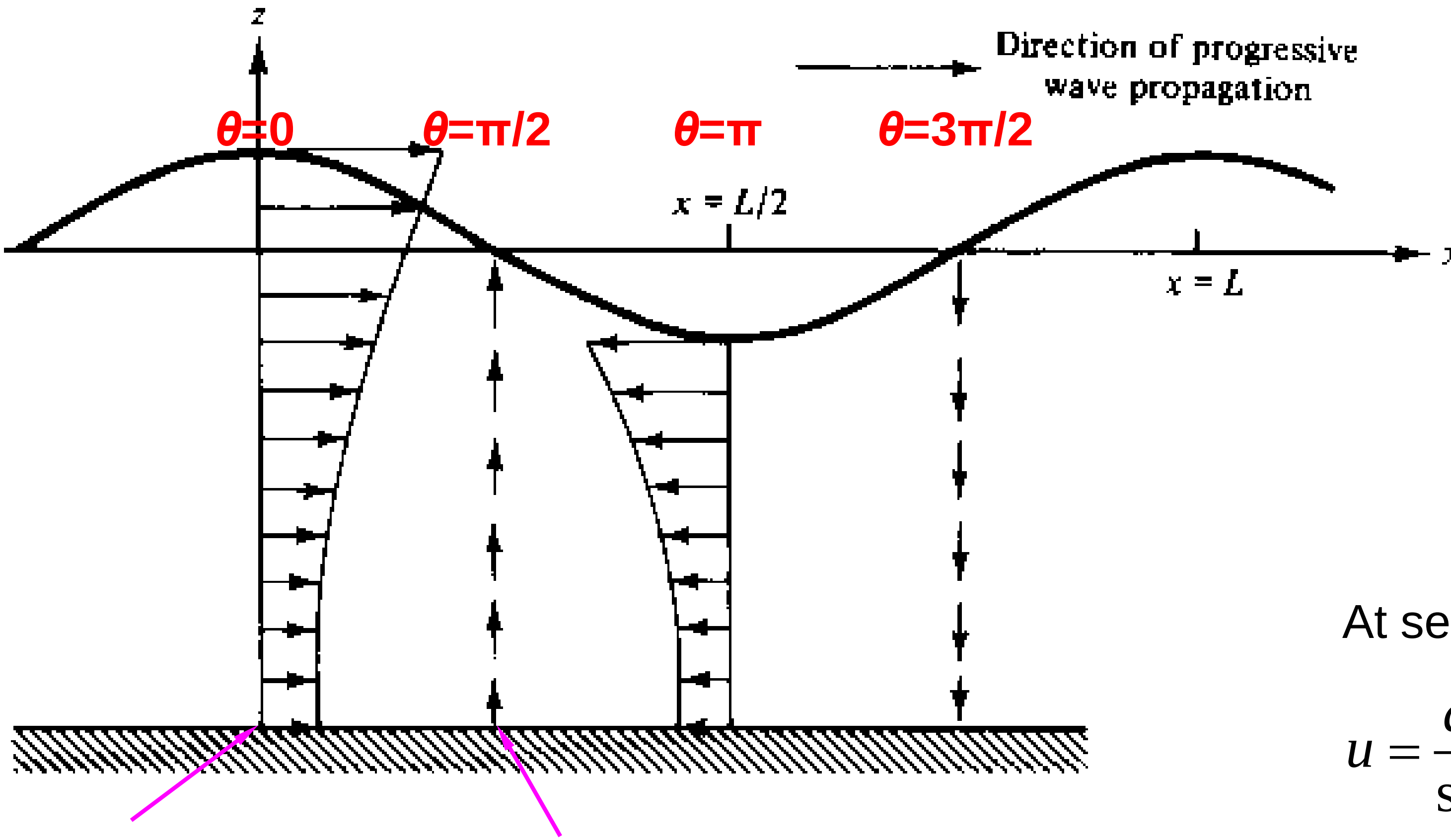
Wave celerity (phase velocity)

Wavelength $$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi h}{L}\right), \qquad C = \frac{gT}{2\pi}\tanh\left(\frac{2\pi h}{L}\right) = \sqrt{\frac{g}{k}\tanh(kh)}$$

$$u = a\sigma\frac{\cosh\left[k(h+z)\right]}{\sinh(kh)}\cos\theta$$ Water particle horizontal velocity

$$w = a\sigma\frac{\sinh\left[k(h+z)\right]}{\sinh(kh)}\sin\theta$$ Water particle vertical velocity

Water particle velocity in a progressive wave



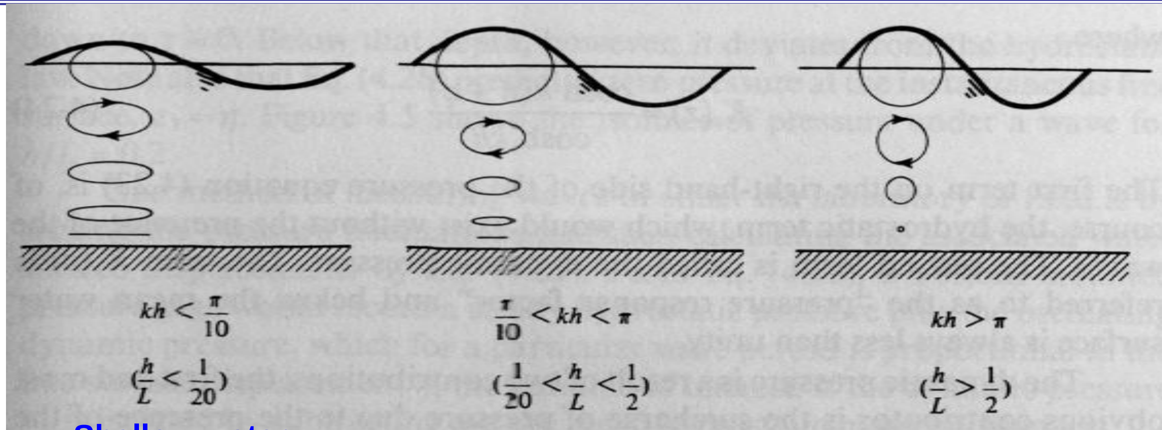
$$u = \frac{a\sigma}{\sinh(kh)}$$

$$w = 0$$

At seabed

$$u = \frac{a\sigma\cos\theta}{\sinh(kh)}$$

$$w = 0$$



Shallow water **Intermediate depth** **Deep water**

water particle trajectories under Airy waves

Wave energy per unit horizontal area

$$E = E_p + E_k = \frac{1}{16}\rho g H^2 + \frac{1}{16}\rho g H^2 = \frac{1}{8}\rho g H^2 = \frac{1}{2}\rho g a^2$$

Potential + Kinetic

For irregular wave: $\overline{E} = \frac{1}{8}\rho g\left(H_{rms}\right)^2 \cong \frac{1}{16}\rho g\left(H_{1/3}\right)^2$

Mean wave energy

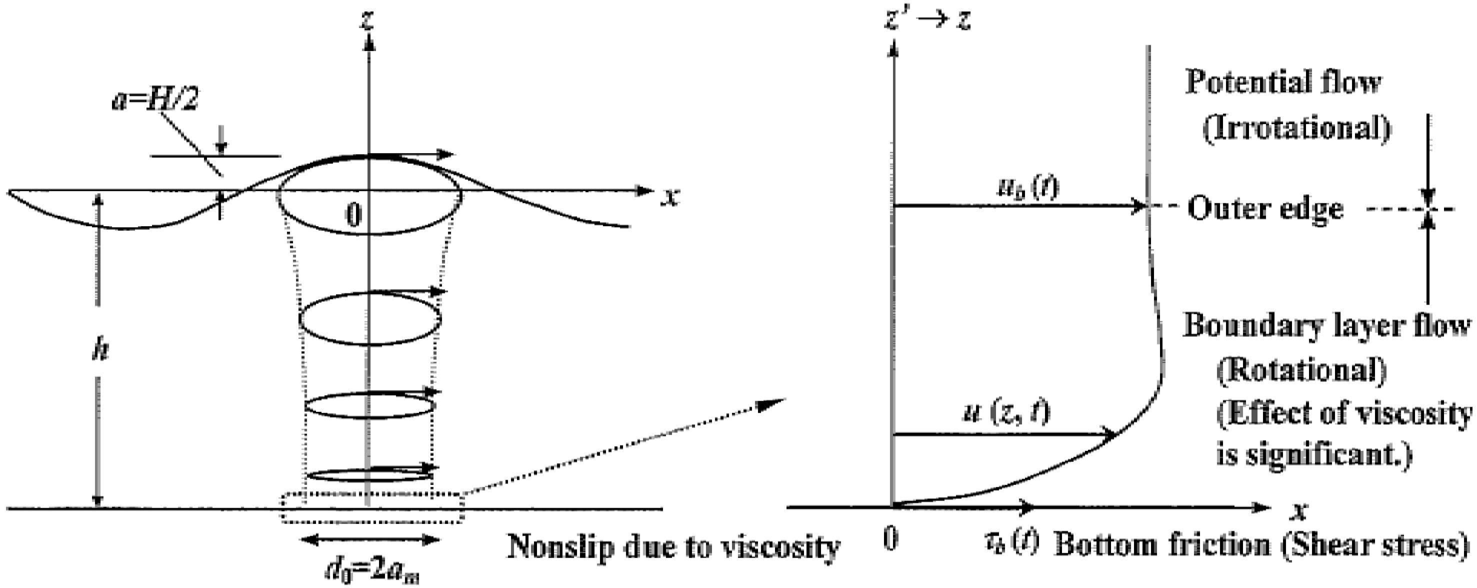
$H_{1/3}$: Significant wave height; root-mean-square w.h. $H_{rms} = \sqrt{\overline{H^2}} \cong 0.707 H_{1/3}$

the highest one-third waves

Energy flux per unit width and unit time

$$F = EC_g;\ \ C_g = nC\ :\ \text{Group velocity}$$

Shallowness parameter $n = \frac{1}{2}\left(1 + \frac{2hk}{\sinh 2hk}\right)$; [deepwater] $\frac{1}{2} < n < 1$ [shallow water]



Definition sketch for wave-induced orbital motion and boundary layer flow

Orbital velocity at the outer edge of boundary layer (*Horizontal harmonic oscillation*)

$$u_b \equiv u\big|_{z=-h} = \frac{\pi H}{T}\frac{1}{\sinh kh}\cos\theta = \hat{u}_b\cos\theta,\ \ \theta \equiv \sigma t - kx$$ **(phase function/angle)**

Near bottom velocity amplitude

$$\hat{u}_b \equiv \frac{\pi H}{T\sinh kh}, \sigma = \frac{2\pi}{T}, a = \frac{H}{2} \Rightarrow \hat{u}_b = \frac{a\sigma}{\sinh kh} = \sigma a_m, \quad a_m \equiv \frac{d_0}{2} = \frac{a}{\sinh kh}$$

a_m : near bottom orbital amplitude; d_0 : orbital diameter (excursion(偏移) length)

Laminar Oscillatory Boundary Layer (small $\hat{u}_b$)

Navier-Stokes' momentum equation in the *x*-direction

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right) \longrightarrow \frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial z^2}$$

Linearization & boundary layer approximation $\frac{\partial}{\partial x} << \frac{\partial}{\partial z}$

On the outer edge $\frac{\partial u_b}{\partial t}=-\frac{1}{\rho}\frac{\partial p_b}{\partial x}$

Very thin boundary layer & *v*≈0 $p \cong p_b$

$$\frac{\partial u}{\partial t}=\frac{\partial u_b}{\partial t}+\nu\frac{\partial^2 u}{\partial z^2}$$

Governing Equation

$$\boxed{\frac{\partial^2 u}{\partial z^2}-\frac{1}{\nu}\frac{\partial u}{\partial t}=-\frac{1}{\nu}\frac{\partial u_b}{\partial t}}$$

Boundary Conditions $u=0\,(z=0);\quad u\rightarrow u_b\,(z\rightarrow\infty)$

$u = \Re\left[f(z)e^{i\theta} \right];\ e^{i\theta} = \cos\theta + i\sin\theta;\ \Re:\text{real part};\quad f(z):\text{complex}$

Governing Eq. ⟹ $\dfrac{d^2 f}{dz^2} - \left(\dfrac{i\sigma}{\nu}\right) f = -\left(\dfrac{i\sigma}{\nu}\right)\hat{u}_b$

B.C. $f = 0(z=0);\ f \to \hat{u}_b(z \to \infty)$

Particular solution: $f = \hat{u}_b$

General solution: $f = Ae^{\pm Dz}$ where $D^2 \equiv i\sigma/\nu \to D = \sqrt{i\sigma/\nu} = (1+i)\beta,\ \beta \equiv \sqrt{\sigma/2\nu}$

f **: to be bounded at *z* →∞** ⟹ $f = \hat{u}_b + Ae^{-(1+i)\beta z}$

B.C. $A = -\hat{u}_b$

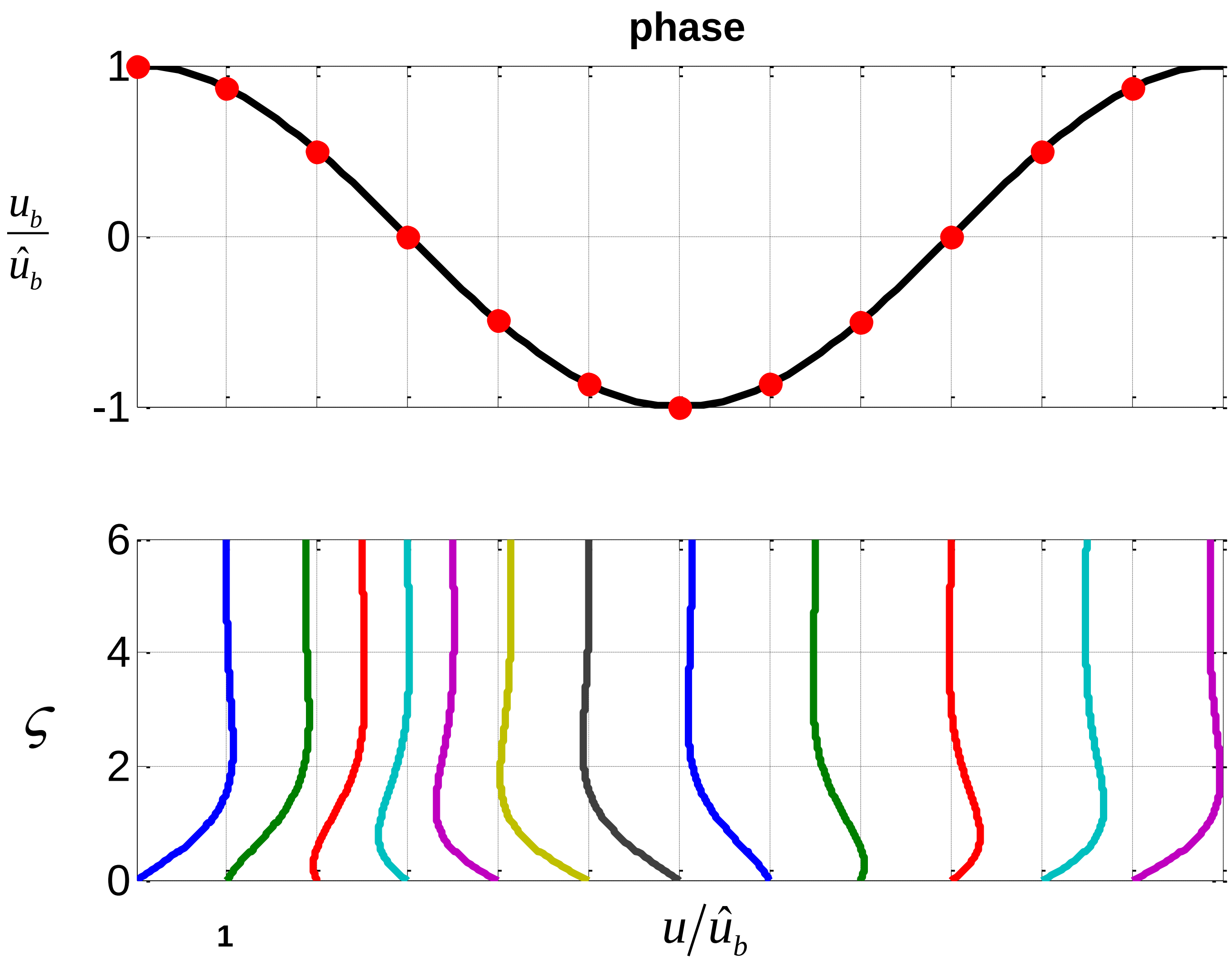
$f = \hat{u}_b\left(1 - e^{-(1+i)\varsigma}\right)$

$\varsigma \equiv \beta z = \dfrac{z}{\delta_l},\ \delta_l \equiv \dfrac{1}{\beta} = \sqrt{\dfrac{2\nu}{\sigma}} = \sqrt{\dfrac{\nu T}{\pi}}$

Stokes layer thickness, become e^{-1}

Solution

$$u = \hat{u}_b \Re\left[e^{i\theta} - e^{i\theta-(1+i)\varsigma} \right]$$
$$= \hat{u}_b\left[\cos\theta - e^{-\varsigma}\cos(\theta - \varsigma) \right]$$



Variation of *u* in a laminar oscillatory boundary layer

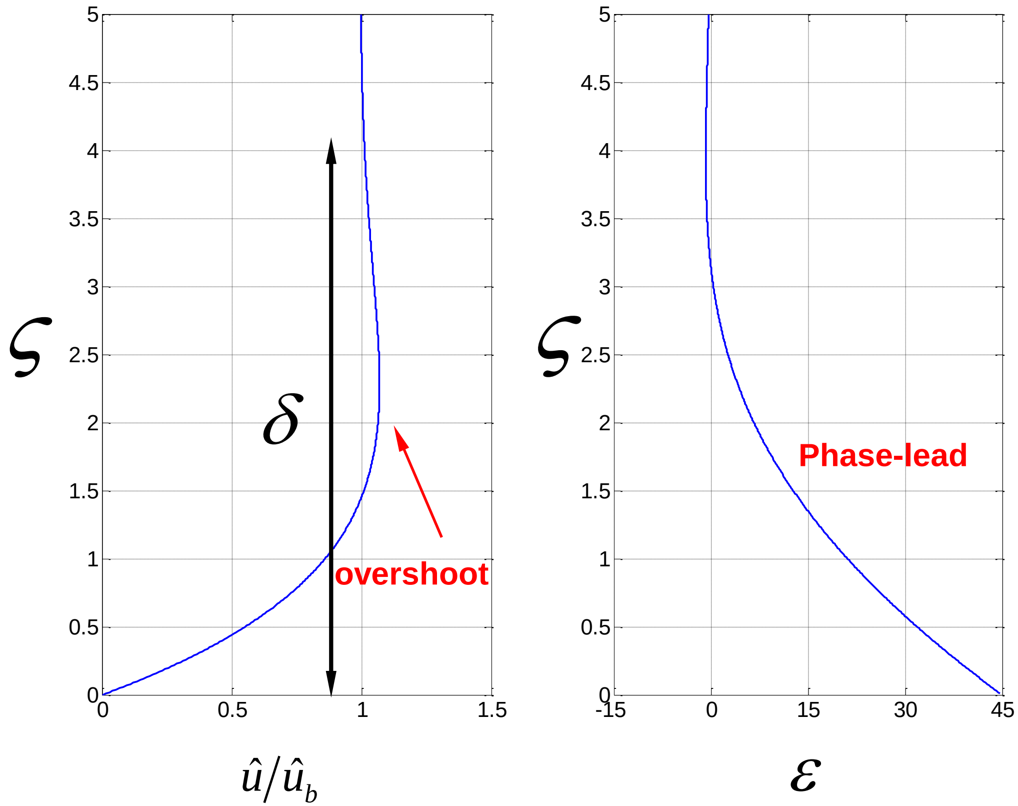
Velocity amplitude $\hat{u}$ and phase shift ε

$$u = \hat{u}\cos(\theta + \varepsilon)$$

$$\hat{u} = \hat{u}_b\sqrt{1 - 2e^{-\varsigma}\cos\varsigma + e^{-2\varsigma}}$$

$$\varepsilon = \tan^{-1}\left[\frac{\sin\varsigma}{e^{\varsigma} - \cos\varsigma}\right]$$

Amplitude and phase shift of ***u*** in a laminar wave boundary layer



Boundary layer thickness δ

$$\frac{\hat{u} - \hat{u}_b}{\hat{u}_b} \cong 1\% \rightarrow \delta = 4.1\delta_l = 4.1\sqrt{\frac{\nu T}{\pi}}$$

$$\delta \cong 2.3\text{mm}(T = 1s),\ 7.3\text{mm}(T = 10s)$$

Bottom friction (*Bottom shear stress*)

$$\tau_b = \mu \frac{\partial u}{\partial z}\bigg|_{z=0} = \mu \hat{u}_b \left(\frac{\cos\theta}{\delta_l} - \frac{\sin\theta}{\delta_l} \right) = \sqrt{2}\mu \frac{\hat{u}_b}{\delta_l} \cos\left(\theta + \frac{\pi}{4}\right) = \hat{\tau}_b \cos\left(\theta + \frac{\pi}{4}\right); \quad \mu = \rho \nu$$

$\hat{\tau}_b \propto \hat{u}_b$ **with phase lead of π/4**

Define "*wave friction factor*" f_w as $\hat{\tau}_b \equiv \frac{1}{2} \rho f_w \hat{u}_b^2$

$$\Rightarrow \quad f_w = \frac{2\hat{\tau}_b}{\rho \hat{u}_b^2} = \frac{2\sqrt{2}\nu}{\hat{u}_b \delta_l} = \frac{2\sqrt{2}}{R_\delta} = \frac{2}{\sqrt{R_a}} \qquad R_\delta \equiv \frac{\hat{u}_b \delta_l}{\nu}, \quad R_a \equiv \frac{\hat{u}_b a_m}{\nu} = \frac{R_\delta^2}{2}$$

$$a_m = \frac{\hat{u}_b}{\sigma}, \delta_l = \sqrt{\frac{\nu T}{\pi}} \qquad a_m \equiv \frac{d_0}{2} = \frac{a}{\sinh kh}$$

Classification of Wave Boundary Layer

1) Laminar boundary layer

Reynolds number R_a

Approximately $R_a > 10^4$ => Laminar to Turbulent

2) Turbulent boundary layer

a) Smooth turbulent boundary layer

b) Rough turbulent boundary layer

$$5 < \frac{\hat{u}_* k_s}{\nu} < 70 \,(\text{Transition})$$

$k_s = 30 z_0$ Nikuradse equivalent roughness, z_0: Roughness length

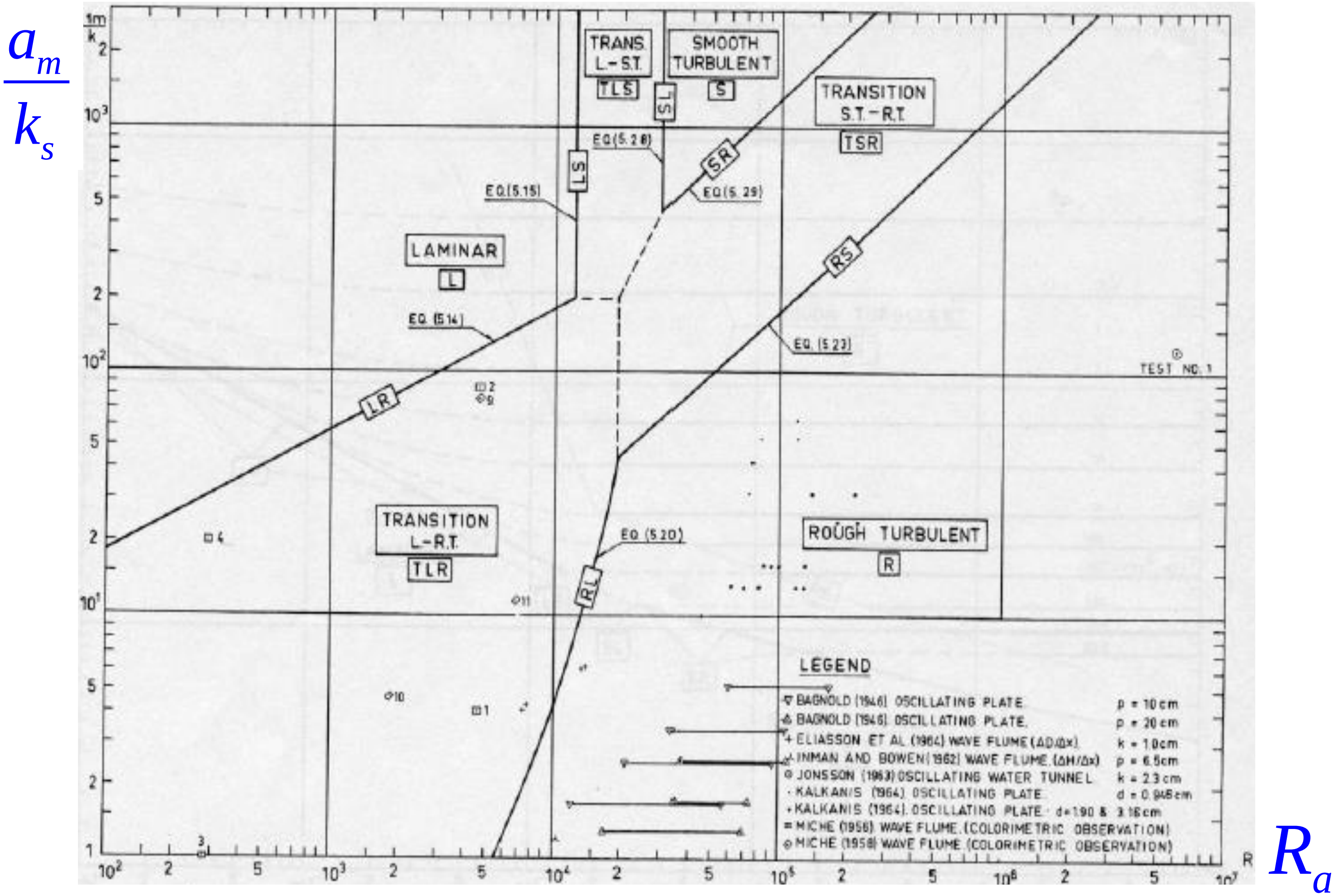
$\hat{u}_* \equiv \sqrt{\hat{\tau}_b / \rho}$ Amplitude of friction velocity

Governing Equation

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial u_b}{\partial t} + \frac{\partial}{\partial z}\left(\frac{\tau}{\rho}\right)}$$

$$\tau = -\rho \overline{u'w'}$$

Reynolds shear stress



Classification of the flow regime (after Jonsson, 1966)

Turbulent Wave Boundary layer and Friction Laws

(1) Jonsson (1966, ICCE, 127): analogy to the steady flow

A) Rough turbulent

B. C. $$\left.\begin{matrix} u=0 \\ \tau=\tau_b \end{matrix}\right\} \text{at } z=z_0\neq 0, \quad \left.\begin{matrix} u=u_b \\ \tau \ll \tau_b \end{matrix}\right\} \text{at } z=\delta_l$$

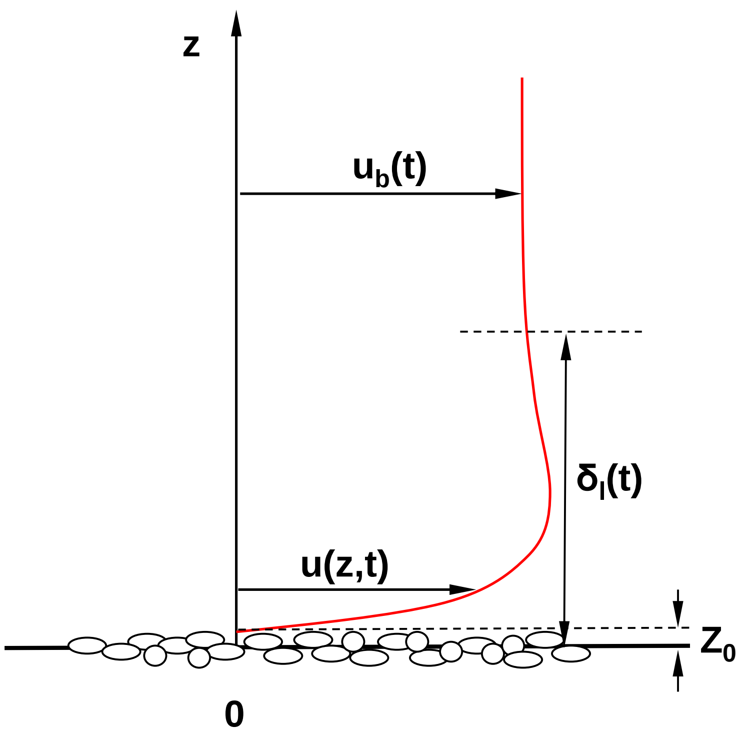
Gov. Eq. ⇨ $$\frac{\partial(u_b-u)}{\partial t}=-\frac{\partial}{\partial z}\left(\frac{\tau}{\rho}\right)$$

Integrated momentum equation

$$\int_{z_0}^{\delta_l}\frac{\partial(u_b-u)}{\partial t}dz=-\frac{\tau}{\rho}\bigg|_{z_0}^{\delta_l}=\frac{\tau_b}{\rho}=u_*^2(t)$$

Assume a logarithmic velocity distribution and neglect the phase shift between *u* and u_*

$$\frac{u(z,t)}{u_*(t)}=\frac{1}{\kappa}\ln\left(\frac{z}{z_0}\right); \qquad \kappa=0.4:\text{Karman's constant}$$



Definition sketch

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x\right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,\mathrm{d}x + f\big(b(t),t\big)\cdot b'(t) - f\big(a(t),t\big)\cdot a'(t)$$

at the upper boundary $$\frac{u_b(t)}{u_*(t)} = \frac{1}{\kappa}\ln\left(\frac{\delta_l}{z_0}\right) \Rightarrow u_* = \frac{\kappa u_b}{\ln(\delta_l/z_0)}$$

$$u_b - u = \frac{u_*}{\kappa}\ln\left(\frac{\delta_l}{z}\right)$$

$$\kappa u_*^2 = \frac{\kappa^3 u_b^2}{\ln^2(\delta_l/z_0)} = \frac{\kappa^3 \hat{u}_b^2 \cos^2 \sigma t}{\ln^2(\delta_l/z_0)}$$

Integrated mom. Eq. $\Rightarrow$ $$u_*^2(t) = \int_{z_0}^{\delta_l} \frac{\partial\left(\frac{u_*}{\kappa}\ln\left(\frac{\delta_l}{z}\right)\right)}{\partial t}dz \Rightarrow \boxed{\kappa u_*^2 \cong \frac{\partial u_*}{\partial t}\delta_l + \frac{\partial \delta_l}{\partial t}u_* = \frac{\partial(u_*\delta_l)}{\partial t}}$$

$z_0 << \delta_l$

$$\Rightarrow u_*\delta_l = \int_{-\pi/(2\sigma)}^{t} \kappa u_*^2 dt \Rightarrow u_*\delta_l = \frac{\kappa^3}{2}\frac{\hat{u}_b^2}{\ln^2(\delta_l/z_0)}\left(t + \frac{\sin 2\sigma t}{2\sigma} + \frac{\pi}{2\sigma}\right)$$

Neglecting the time variation of δ_l

Representative boundary layer thickness $\delta_T = \delta_l(\text{at } \sigma t = 0, \text{ when } u_b(t) = \hat{u}_b)$

$$\Rightarrow \hat{u}_*\delta_T = \frac{\kappa^3}{2}\frac{\hat{u}_b^2}{\ln^2(\delta_T/z_0)}\frac{\pi}{2\sigma} = \frac{\pi\kappa^3}{4}\frac{\hat{u}_b a_m}{\ln^2(\delta_T/z_0)}$$

$\hat{u}_b = a_m\sigma$

$$\frac{\delta_T}{z_0}\ln\left(\frac{\delta_T}{z_0}\right)=\frac{\pi\kappa^2}{4}\frac{a_m}{z_0} \Rightarrow \ln\left(\frac{\delta_T}{z_0}\right)+\ln\left[\ln\left(\frac{\delta_T}{z_0}\right)\right]=\ln\left(\frac{\pi\kappa^2}{4}\right)+\ln\left(\frac{a_m}{z_0}\right)$$

In case of friction factor

$$f_w=\frac{2}{\rho}\frac{\hat{\tau}_b}{\hat{u}_b^2}=2\left(\frac{\hat{u}_*}{\hat{u}_b}\right)^2=\frac{2\kappa^2}{\ln^2\left(\delta_T/z_0\right)} \Rightarrow \ln\left(\frac{\delta_T}{z_0}\right)=\frac{\sqrt{2}\kappa}{\sqrt{f_w}}$$

$z_0=k_s/30$

$$\frac{\sqrt{2}\kappa}{\sqrt{f_w}}+\ln\left(\frac{\sqrt{2}\kappa}{\sqrt{f_w}}\right)=\ln\left(\frac{\pi\kappa^2}{4}\right)+\ln\left(\frac{30a_m}{k_s}\right)$$

$\div 4\sqrt{2}\kappa$

$$\frac{1}{4\sqrt{f_w}}+\frac{1}{4\sqrt{2}\kappa}\ln\left(\frac{\sqrt{2}\kappa}{\sqrt{f_w}}\right)=\frac{1}{4\sqrt{2}\kappa}\ln\left(\frac{\pi\kappa^2}{4}\times 30\right)+\frac{1}{4\sqrt{2}\kappa}\ln\left(\frac{a_m}{k_s}\right)$$

$$\kappa=0.4,\ \ln x\cong 2.303\log_{10}x \rightarrow \frac{1}{4\sqrt{2}\kappa}\ln x\cong 1.018\log_{10}x\cong\log_{10}x$$

$$\frac{1}{4\sqrt{f_w}}+\log_{10}\frac{1}{4\sqrt{f_w}}=\log_{10}\left(\frac{a_m}{k_s}\right)+0.22$$

Modification based on measurements (through only few data)
cf. Bagnold's data, f_w=0.30 (a_m/k_s<1.57)

$$\frac{1}{4\sqrt{f_w}}+\log_{10}\frac{1}{4\sqrt{f_w}}=\log_{10}\left(\frac{a_m}{k_s}\right)-0.08$$

Implicit formula

Explicit approximation to Jonsson (1966) for rough turbulence case, Swart (1974)

$$\begin{cases} f_w = \exp\left[-5.997 + 5.213\left(\dfrac{k_s}{a_m}\right)^{0.194}\right] & \dfrac{k_s}{a_m} < 0.63 \\ f_w = 0.3 & \dfrac{k_s}{a_m} \geq 0.63 \end{cases}$$

B) Smooth turbulent

Replace z_0 by a parameter, i.e., the viscous sublayer thickness $\delta_v \equiv 11.6\nu/\hat{u}_*$

Namely, $z_0 = \dfrac{k_s}{30} = \dfrac{\delta_v}{105} \rightarrow k_s = \dfrac{3.33\nu}{\hat{u}_*} = \dfrac{3.33\nu}{\sqrt{f_w/2}\hat{u}_b}$ $\qquad R_a = \dfrac{\hat{u}_b a_m}{\nu}$

$$\frac{1}{4\sqrt{f_w}} + 2\log_{10}\frac{1}{4\sqrt{f_w}} = \log_{10} R_a - 1.36$$

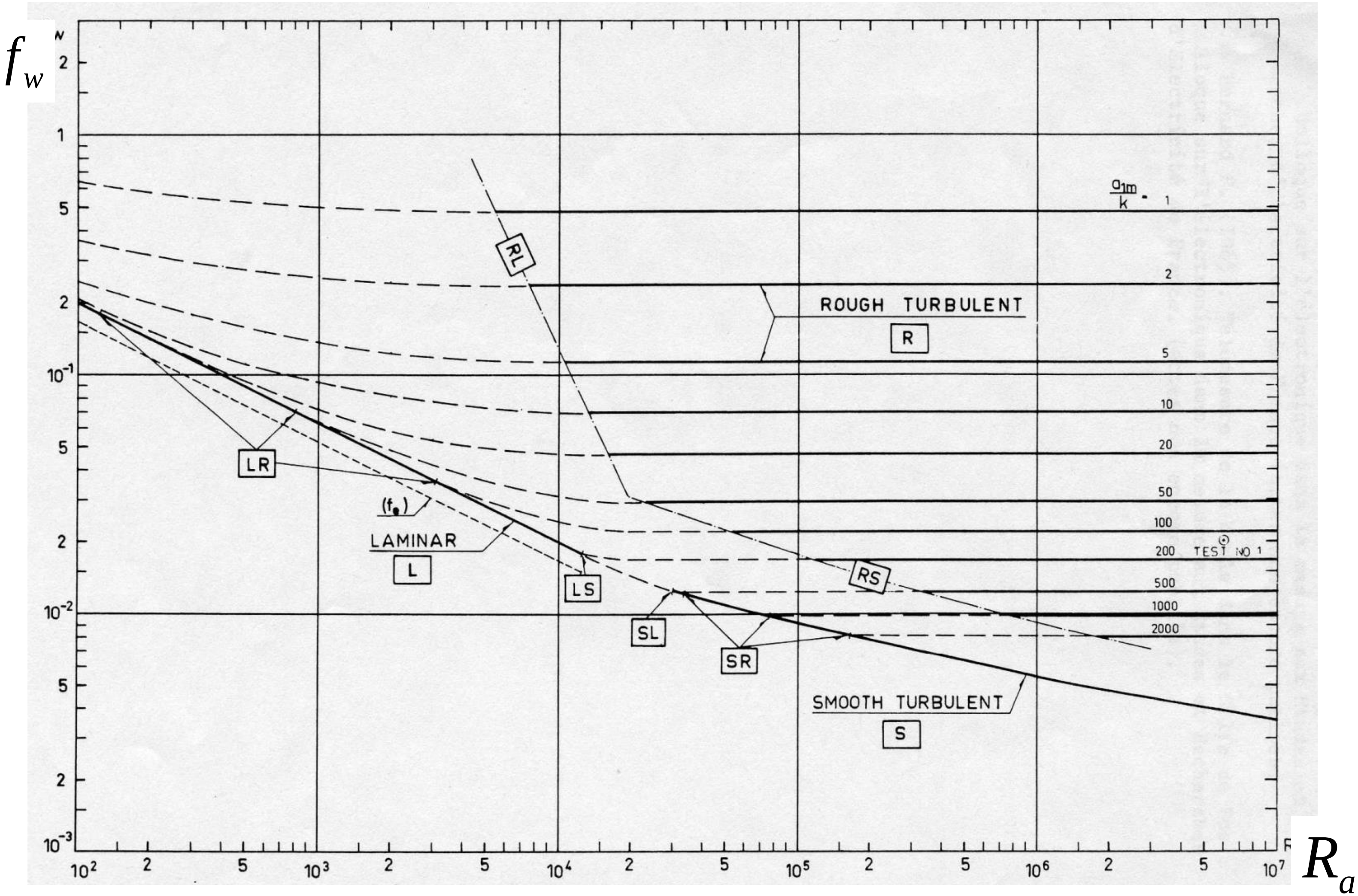
$$z_0 = \frac{k_s}{30} = \frac{\delta_v}{105} \rightarrow k_s = \frac{3.33\nu}{\hat{u}_*} = \frac{3.33\nu}{\sqrt{f_w/2}\hat{u}_b}$$

Modification

$$\frac{1}{4\sqrt{f_w}} + 2\log_{10}\frac{1}{4\sqrt{f_w}} = \log_{10} R_a - 1.55$$

Explicit form

$$f_w = 0.09 R_a^{-0.2}$$



Wave friction factor (after Jonsson, 1966)

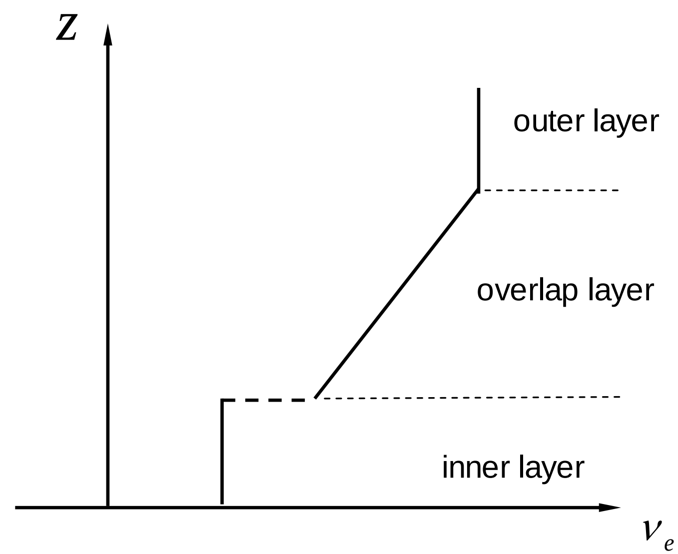
(2) Kajiura (1968, Bull. Earthquake Res. Inst.. Univ. of Tokyo, Vol 46, 75)

Gov. Eq. ⟹ $$\frac{\partial u}{\partial t}=\frac{\partial u_b}{\partial t}+\frac{\partial}{\partial z}\left(\frac{\tau}{\rho}\right)=\frac{\partial u_b}{\partial t}+\frac{\partial}{\partial z}\left(\nu_e\frac{\partial u}{\partial z}\right)$$

Kinematic eddy viscosity ν_e: Three-layer model

A) Smooth turbulent

$$\nu_e=\begin{cases}\nu & (0\le z\le\delta_i):\text{inner layer}\\ \kappa\hat{u}_* z & (\delta_i\le z\le\delta_o):\text{overlap layer}\\ K\hat{u}_b\delta^* & (\delta_o\le z):\text{outer layer}\end{cases}$$



Schematic of ν_e distribution

δ_i Inner layer (viscous sublayer) thickness $=12\nu/\hat{u}_*\cong\delta_V$

δ_o wall layer thickness $\leftarrow\kappa\hat{u}_*\delta_o=K\hat{u}_b\delta^*$

$$K=0.02;\ \delta^*\equiv\mathrm{Amp}\left(\int_0^\infty\frac{u_b-u}{\hat{u}_b}dz\right):\text{displacement thickness}$$

B) Rough turbulent

$$\nu_e = \begin{cases} 0.185\kappa \hat{u}_* k_s & \left(0 \le z \le k_s/2\right): \text{inner layer} \\ \kappa \hat{u}_* z & \left(k_s/2 \le z \le \delta_o\right): \text{overlap layer} \\ \kappa \hat{u}_* \delta_o & \left(\delta_o \le z\right): \text{outer layer} \end{cases}$$

Boundary condition $\begin{cases} u = 0(z=0);\ u = u_b\ (z \to \infty) \\ u\ \&\ \partial u/\partial z: \text{continuous at the two interfaces} \end{cases}$

Very complicated analytical solutions with Bessel functions of complex variables

"Complex friction factor" $\tilde{C}$ $\begin{cases} \tau_b = \hat{\tau}_b e^{i(\theta+\varepsilon)} \equiv \rho \tilde{C} \hat{u}_b u_b,\ \tilde{C} = \hat{C} e^{i\varepsilon} \\ \hat{\tau}_b = \rho \hat{C} \hat{u}_b^2 \qquad \rightarrow \hat{C} = f_w/2 \end{cases}$

$$R \equiv R_\delta/\sqrt{2} = \sqrt{R_a},\ \ \hat{u}_b/(\sigma z_0) = a_m/z_0 = 30\left(a_m/k_s\right)$$

Approximated expressions:

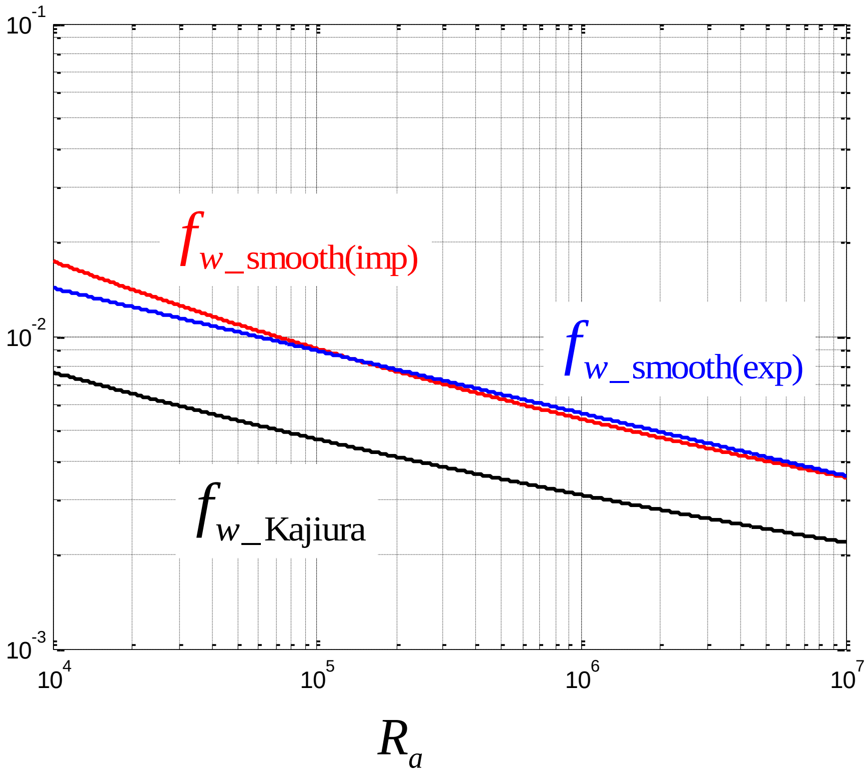
Smooth

$$\frac{1}{4.05\sqrt{f_w}} + 2\log_{10}\frac{1}{4\sqrt{f_w}} = \log_{10} R_a - 0.27$$

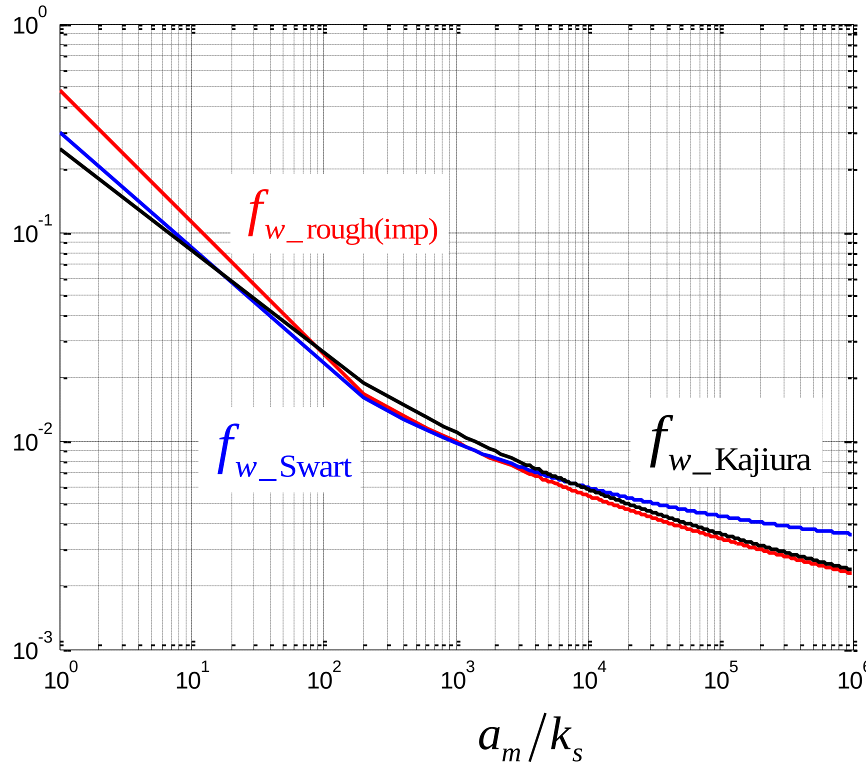
Rough

$$\begin{cases} \dfrac{1}{4.05\sqrt{f_w}} + \log_{10}\dfrac{1}{4\sqrt{f_w}} = \log_{10}\left(\dfrac{a_m}{k_s}\right) - 0.254 \\ f_w = 0.25 \quad \left(a_m/k_s < 1.67\right) \end{cases}$$

Smooth turbulence



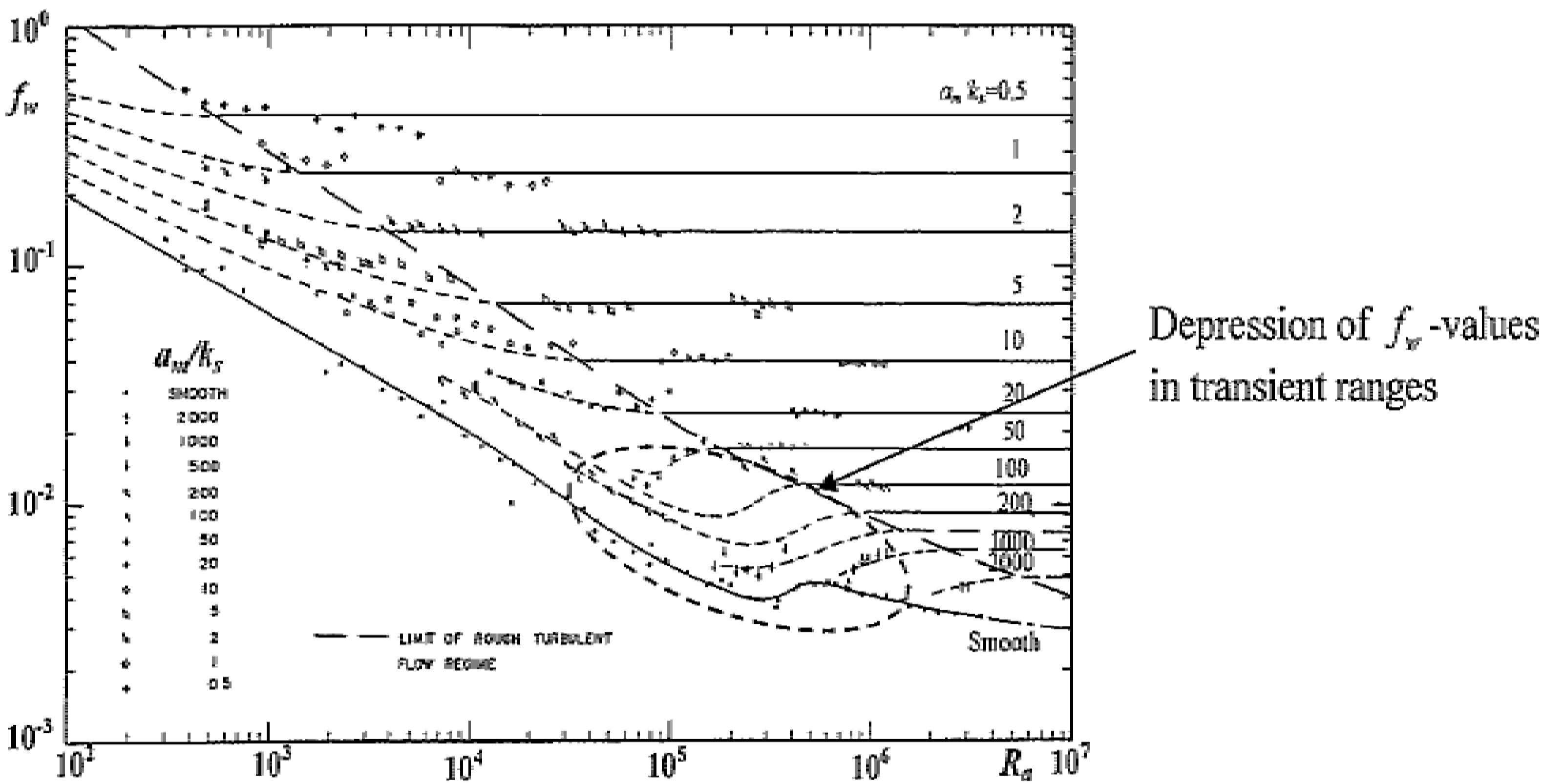
Rough turbulence



Wave friction factor for smooth and rough turbulence (implicit & explicit)

(3) Riedel, Kamphuis & Brebner (1972,13th ICCE, 587)

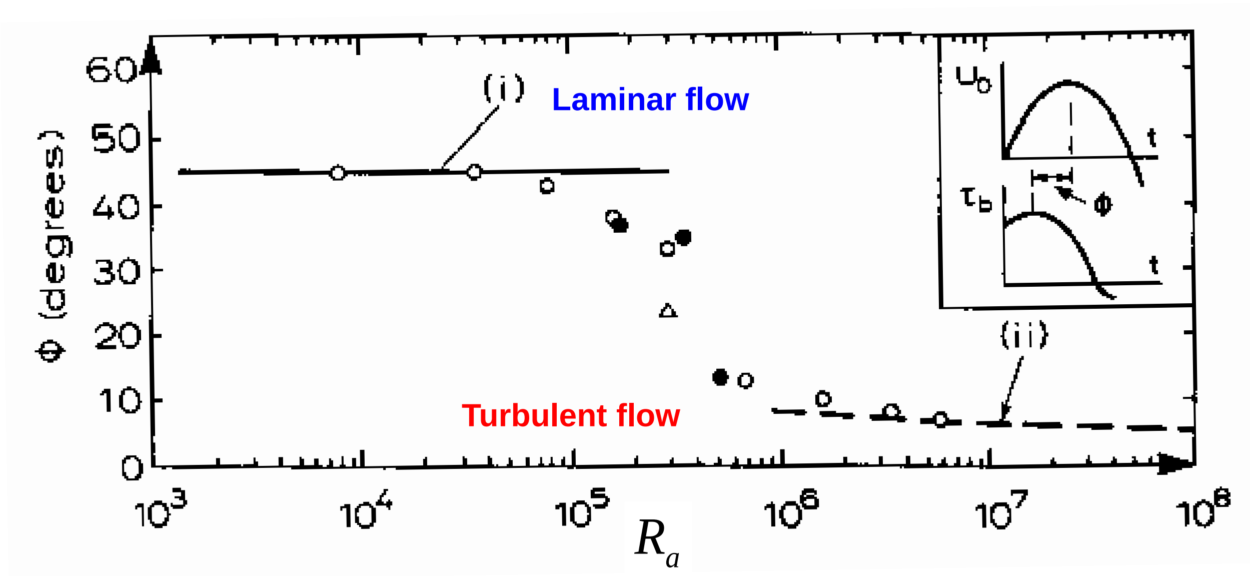
Direct measurement of τ_b in an oscillatory flow tank using a shear meter



Measured wave friction factor

Phase lead is small in case of turbulent flows, ~ 10^0

Phase difference between bed shear stress and free-stream velocity
(Fredsøe & Deigaard, 1992)



(4) US CERC (1984):

$$\tau_{\mathrm{w}} = \frac{1}{2}\rho f_{\mathrm{w}} U_{\mathrm{b}}^{2} \qquad f_{\mathrm{w}} = 2\left(\frac{U_{\mathrm{b}} A_{\mathrm{b}}}{\nu}\right)^{-0.5}$$

$$A_{\mathrm{b}} = \frac{1}{2\sinh(2\pi d/L)}$$

$$U_{\mathrm{b}} = \frac{\pi H}{T\sinh(2\pi d/L)}$$

A_b: maximum bottom wave orbital amplitude
U_b: maximum bottom wave orbital velocity
d: water depth

$$\tau_{\mathrm{w}} = H\left[\rho \frac{\left(\nu\left(\frac{2\pi}{T}\right)^{3}\right)^{0.5}}{2\sinh\left(\frac{2\pi d}{L}\right)}\right]$$

Friction Factor in a Coexistent Wave-current Field

(1) Grant & Madsen (1979, *J. Geo. Res.*, 84, 1979)

$$f_{cw} = \alpha f_c + (1-\alpha) f_w, \quad \alpha = \frac{U_c}{\hat{u}_b + U_c}$$

for the collinear case

Logarithmic velocity distribution $$f_c = 2\kappa^2 \left[\ln\left(\frac{z_{um}}{z_0} \right) \right]^{-2}$$

Z_{um}: where the current velocity *Uc* is expected
Z_0=k_s/30: where the current velocity is zero

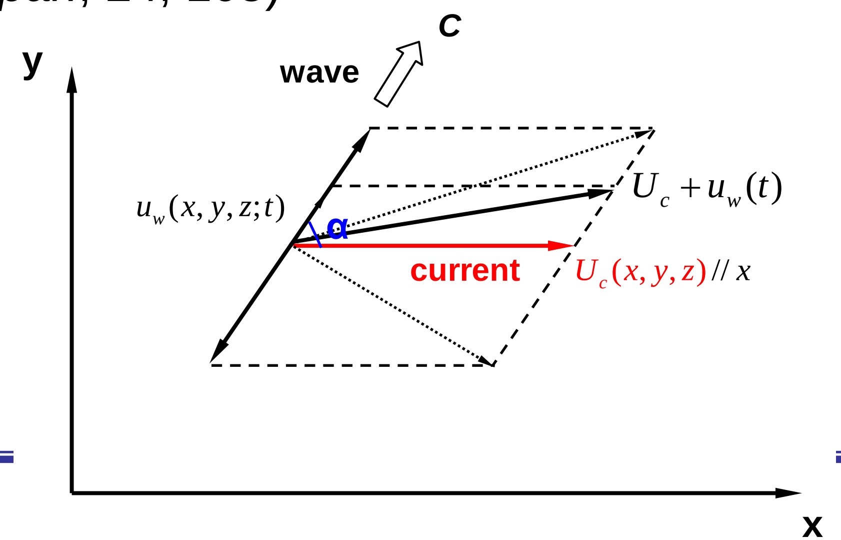
The first work presents an analytical solution for the wave-current friction factor

(2) Tanaka & Shuto (1981, *Coastal Eng. in Japan*, 24, 105)

One-layer model for v_e

Orbital velocity amplitude
at the outer edge of the boundary layer

$$\hat{\mathbf{u}}_b = \frac{\pi H}{T \sinh(2\pi h / L)} (\cos\alpha, \sin\alpha)$$



Definition sketch of a wave-current coexistent system

Moving coordinate system with velocity $\bar{U}_c \cos\alpha$

No apparent current in wave direction & Same L → wave celerity $C \rightarrow C-\bar{U}_c\cos\alpha \downarrow$; then, $T \rightarrow T' \uparrow$

$$L = CT = \left(C-\bar{U}_c\cos\alpha\right)T' \rightarrow \frac{1}{T'} = \frac{1}{L}\left(\frac{L}{T}-\bar{U}_c\cos\alpha\right)$$

$$\hat{\mathbf{u}}_b = \frac{\pi H}{L}\frac{L/T-\bar{U}_c\cos\alpha}{\sinh\left(2\pi h/L\right)}\left(\cos\alpha,\sin\alpha\right)$$

Rough turbulent (example)

Boundary layer equation

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial\left(\nu_e\,\partial u/\partial z\right)}{\partial z} \qquad \mathbf{u}(z,t)=\mathbf{u}_w(z,t)+\mathbf{U}_c(z)$$

Keeping wave-current combined effect through ν_e

1 $$\frac{\partial \mathbf{u}_w}{\partial t} = -\frac{1}{\rho}\nabla p_w+\frac{\partial\left(\nu_e\,\partial \mathbf{u}_w/\partial z\right)}{\partial z}$$

2 $$\frac{\partial\left(\nu_e\,\partial \mathbf{U}_c/\partial z\right)}{\partial z} = \frac{1}{\rho}\nabla p_c \cong 0; \qquad \mathbf{U}_c=\left(U_c,0\right)$$

At the outer edge of the boundary $\frac{\partial \mathbf{u}_b}{\partial t} = -\frac{1}{\rho}\nabla p_{wb} \cong -\frac{1}{\rho}\nabla p_w$ thin boundary layer

One layer model for eddy viscosity $\nu_e = \kappa\hat{u}^*_{cw}z,\ \kappa=0.4,\ \hat{u}^*_{cw} \equiv \sqrt{\left|\boldsymbol{\tau}_{b\max}/\rho\right|}$ Friction velocity amplitude

2 ⇨ $$\nu_e \frac{dU_c}{dz} = const. = \frac{\tau_c}{\rho} \equiv u_c^{*2} \rightarrow \frac{dU_c}{dz} = \frac{u_c^{*2}}{\kappa \hat{u}_{cw}^* z} \Rightarrow U_c = \frac{u_c^{*2}}{\kappa \hat{u}_{cw}^*} \ln\left(\frac{z}{z_0}\right)$$

$$\int \ln x dx = x(\ln x - 1)$$

⇨ $$\int_{z_0}^{h} U_c dz = \bar{U}_c (h - z_0) = \frac{u_c^{*2}}{\kappa \hat{u}_{cw}^*}\left[z\left\{\ln\left(\frac{z}{z_0}\right) - 1\right\}\right]_{z_0}^{h} = \frac{u_c^{*2}}{\kappa \hat{u}_{cw}^*}\left[h \ln\left(\frac{h}{z_0}\right) - h + z_0\right]$$

⇨ $$u_c^{*2} = \frac{\kappa \bar{U}_c \hat{u}_{cw}^*}{\frac{1}{1 - z_0/h}\ln\left(\frac{h}{z_0}\right) - 1} \cong \frac{\kappa \bar{U}_c \hat{u}_{cw}^*}{\ln\left(\frac{h}{z_0}\right) - 1}$$

1 ⇨ $$\frac{\partial(\mathbf{u}_w - \mathbf{u}_b)}{\partial t} = \frac{\partial}{\partial z}\left(\kappa \hat{u}_{cw}^* z \frac{\partial \mathbf{u}_w}{\partial z}\right)$$

Considering the motion is periodic in time $\mathbf{u}_b - \mathbf{u}_w \equiv \mathbf{v} = \hat{v}(z) e^{i\sigma t}$ $\hat{v}(z)$ complex function

$$-i\sigma \hat{v} e^{i\sigma t} = \kappa \hat{u}_{cw}^*\left[-\frac{d\hat{v}}{dz} - z\frac{d^2\hat{v}}{dz^2}\right] e^{i\sigma t} \Rightarrow z\frac{d^2\hat{v}}{dz^2} + \frac{d\hat{v}}{dz} - ic\hat{v} = 0;\ c \equiv \frac{\sigma}{\kappa \hat{u}_{cw}^*}$$

Introduce a new complex variable

$$\xi \equiv 2\sqrt{cz}\, e^{-i\pi/4}$$

⇨ $$\boxed{\xi^2 \frac{d^2\hat{v}}{d\xi^2} + \xi\frac{d\hat{v}}{d\xi} + \xi^2 \hat{v} = 0 \rightarrow \frac{d^2\hat{v}}{d\xi^2} + \frac{1}{\xi}\frac{d\hat{v}}{d\xi} + \hat{v} = 0}$$

Bessel's differential equation of the zero-th order

Bessel's differential equation of the γ-th order $$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{\gamma^2}{x^2}\right)y=0$$

0-th order Bessel's D.E. ⇨ General solution $\hat{v}=C_1J_0(\xi)+C_2N_0(\xi)$

$$J_\gamma=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\gamma+n+1)}\left(\frac{x}{2}\right)^{2n+\gamma}$$ First kind of Bessel function

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx\ \ (\alpha>0)$$ Gama function

$$N_\gamma=\frac{\cos\gamma\pi J_\gamma-J_{-\gamma}}{\sin\gamma\pi}$$ Second kind of Bessel function

B.C.
- No-slip $\mathbf{u}_w=0\,(z=z_0)\rightarrow\hat{v}=\hat{\mathbf{u}}_b\left(\xi=\xi_0\right)$
- Shear-free $\boldsymbol{\tau}_w=0(z=h)\rightarrow d\hat{v}/d\xi=0\left(\xi=\xi_h\right)$

$$J_0'=-J_1 \qquad N_0'=-N_1 \Rightarrow \begin{cases}\hat{\mathbf{u}}_b=C_1J_0(\xi_0)+C_2N_0(\xi_0)\\ 0=C_1J_1(\xi_h)+C_2N_1(\xi_h)\end{cases}\rightarrow C_1=\frac{\hat{\mathbf{u}}_b}{M}N_1(\xi_h),\ C_2=-\frac{\hat{\mathbf{u}}_b}{M}J_1(\xi_h)$$

$$M\equiv N_1(\xi_h)J_0(\xi_0)-J_1(\xi_h)N_0(\xi_0)$$

⇨ $$\mathbf{u}_w=\mathbf{u}_b-\mathbf{v}=\hat{\mathbf{u}}_b\Re\left\{\left[1-\frac{N_1(\xi_h)J_0(\xi)-J_1(\xi_h)N_0(\xi)}{N_1(\xi_h)J_0(\xi_0)-J_1(\xi_h)N_0(\xi_0)}\right]e^{i\sigma t}\right\}$$

Bottom friction (shear stress)

$$\xi \equiv 2\sqrt{cz}e^{-i\pi/4},\ c = \frac{\sigma}{\kappa \hat{u}_{cw}^*}$$

$$\boldsymbol{\tau}_b = \rho\kappa\hat{u}_{cw}^* z \left.\frac{\partial \mathbf{u}}{\partial z}\right|_{z=z_0} = \left(\rho u_c^{*2}, 0\right) + \rho\sqrt{\kappa\hat{u}_{cw}^*\sigma z_0}\,|\hat{\mathbf{u}}_b|\,\Re\left[Ae^{i(\sigma t+\varepsilon)}\right](\cos\alpha, \sin\alpha)$$

$$Ae^{i\varepsilon} = \frac{N_1(\xi_h)J_1(\xi_0) - J_1(\xi_h)N_1(\xi_0)}{N_1(\xi_h)J_0(\xi_0) - J_1(\xi_h)N_0(\xi_0)}e^{-i\pi/4} \Leftarrow \mathbf{u}_w = \hat{\mathbf{u}}_b\Re\left\{\left[1 - \frac{N_1(\xi_h)J_0(\xi) - J_1(\xi_h)N_0(\xi)}{N_1(\xi_h)J_0(\xi_0) - J_1(\xi_h)N_0(\xi_0)}\right]e^{i\sigma t}\right\}$$

Maximum bottom friction

$$|\boldsymbol{\tau}_{b\max}| = \tau_{b\max} = \rho\hat{u}_{cw}^{*2} = \left\{\left(\rho u_c^{*2}\right)^2 + 2\rho u_c^{*2}\rho\sqrt{\kappa\hat{u}_{cw}^*\sigma z_0}\,|\hat{\mathbf{u}}_b|\,A\cos\alpha + \rho^2\kappa\hat{u}_{cw}^*\sigma z_0\hat{\mathbf{u}}_b^2A^2\right\}^{1/2}$$

$$\Big\Downarrow \frac{(\ )^2}{\hat{u}_w^2\hat{u}_{cw}^{*2}}$$

Considering expression of u_c^* and $\hat{u}_w \equiv |\hat{\mathbf{u}}_b| = \hat{u}_b$

$$\boxed{u_c^{*2} = \frac{\kappa\bar{U}_c\hat{u}_{cw}^*}{\ln(h/z_0) - 1}}$$

$$\left(\frac{\hat{u}_{cw}^*}{\hat{u}_w}\right)^2 = \left\{\frac{\kappa}{\ln(h/z_0)-1}\right\}^2\left(\frac{\bar{U}_c}{\hat{u}_w}\right)^2 + \frac{2\kappa}{\ln(h/z_0)-1}\left(\frac{\kappa z_0\sigma}{\hat{u}_{cw}^*}\right)^{1/2}A\frac{\bar{U}_c}{\hat{u}_w}\cos\alpha + \frac{\kappa z_0\sigma}{\hat{u}_{cw}^*}A^2$$

Define "wave-current friction factor" f_{cw} by

$$\tau_{b\max} = \rho\hat{u}_{cw}^{*2} \equiv \frac{\rho}{2}f_{cw}\hat{u}_w^2 \Rightarrow \left(\frac{\hat{u}_{cw}^*}{\hat{u}_w}\right)^2 = \frac{f_{cw}}{2}$$

$$\frac{\sigma z_0}{\hat{u}^*_{cw}} = \frac{\sqrt{2/f_{cw}}}{a_m/z_0} \rightarrow \text{RHS} = \text{Function of}\left(\bar{U}_c/\hat{u}_w, \alpha, h/z_0, a_m/z_0, f_{cw}\right)$$

$$\Rightarrow \boxed{f_{cw} = f_{cw}\left(\bar{U}_c/\hat{u}_w, \alpha, h/a_m, a_m/z_0\right)}$$

For "Laminar" and "Smooth turbulent"

$$\boxed{f_{cw} = f_{cw}\left(\bar{U}_c/\hat{u}_w, \alpha, h/a_m, R_a = \frac{\hat{u}_w a_m}{\nu}\right)}$$

Wave-Current Friction Law Covering all Flow Regimes

Tanaka & Thu (1993) and Tanaka & Sana (1996)

Approximated explicit expressions for Tanaka & Shuto (1981)

$$\tau_{b\max} = \rho \hat{u}^{*2}_{cw} = \frac{\rho}{2} f_{cw} \hat{u}^2_w \quad \left(u_w = \hat{u}_w = \hat{u}_b, U_c = \bar{U}_c\right)$$

$$\alpha' = \cos^{-1}\left(\left|\cos\alpha\right|\right) \text{ (rad) } \left(0 \le \alpha' \le \pi/2, 0 \le \alpha \le 2\pi\right)$$

Laminar:

$$f_{cw(L)} = \left\{ f_{c(L)}^2 + 2 f_{c(L)} f_{w(L)} \cos\alpha' + f_{w(L)}^2 \right\}^{1/2}$$

$$f_{c(L)} = \frac{6}{R_c}\left(\frac{U_c}{u_w}\right)^2,\ f_{w(L)} = \frac{2}{\sqrt{R_a}},\ R_c = \frac{U_c h}{\nu},\ R_a = \frac{u_w a_m}{\nu},\ \varepsilon_{(L)} = \pi/4$$

Smooth Turbulent: $(U_c/u_w) \le 5.0$

$$f_{cw(S)} = f_{c(S)} + 2\sqrt{f_{c(S)}\beta_{(S)} f_{w(S)}}\cos\alpha' + \beta_{(S)} f_{w(S)}$$

$$f_{c(S)} = \exp\left\{-7.60 + 5.98 R_c^{-0.0977}\right\}\left(\frac{U_c}{u_w}\right)^2$$

$$f_{w(S)} = \exp\left\{-7.94 + 7.35 R_a^{-0.0748}\right\}$$

$$\beta_{(S)} = \frac{1 + 0.871 R_c^{-0.0362} f_{c(S)}^{0.177} \left(2\alpha'/\pi\right)^{2.5}}{1 + 5.04 R_c^{-0.0303} f_{c(S)}^{0.379}}$$

$$\varepsilon_{(S)} = 0.74\xi_{(S)}^{0.153} \frac{1 + 0.00279\xi_{(S)}^{-0.357}}{1 + 0.127\xi_{(S)}^{0.563}}$$

$$\xi_{(S)} = 0.111 \Big/ \left\{\kappa f_{cw(S)} R_a/2\right\}$$

Rough Turbulent:

$$f_{cw(R)} = f_{c(R)} + 2\sqrt{f_{c(R)}\beta_{(R)} f_{w(R)}}\cos\alpha' + \beta_{(R)} f_{w(R)}$$

$$f_{c(R)} = \frac{2\kappa^2}{\left[\ln\left(h/z_0\right) - 1\right]^2}\left(\frac{U_c}{u_w}\right)^2$$

$$f_{w(R)} = \exp\left\{-7.53 + 8.07\left(\frac{a_m}{z_0}\right)^{-0.1}\right\}$$

$$\beta_{(R)} = \frac{1 + 0.863\beta_1 \exp\left(-1.43\beta_1\right)\left(2\alpha'/\pi\right)^2}{1 + 0.769\beta_1^{0.83}}$$

$$\beta_1 = \frac{1}{\ln\left(h/z_0\right) - 1}\frac{U_c}{u_w},\quad z_0 = \frac{k_s}{30}$$

$$\varepsilon_{(R)} = 0.74\xi_{(R)}^{0.153}\frac{1 + 0.00279\xi_{(R)}^{-0.357}}{1 + 0.127\xi_{(R)}^{0.563}},\quad \xi_{(R)} = 1 \Big/ \left\{\kappa\sqrt{\frac{f_{cw(R)}}{2}}\frac{a_m}{z_0}\right\}$$

All Flow Regimes

$$f_{cw} = f_2\left\{ f_1 f_{cw(L)} + (1-f_1) f_{cw(S)} \right\} + (1-f_2) f_{cw(R)}$$

$$f_1 = \exp\left\{ -0.0513\left(\frac{R_{cw}}{2.5\times10^5} \right)^{4.65} \right\},\quad f_2 = \exp\left\{ C_1\left(\frac{R_{cw}}{R_1} \right)^{C_2} \right\}$$

$$C_1 = -0.0101 - 0.3469\gamma^{0.2},\quad C_2 = 2.06 - 1.09\gamma^{0.05}$$

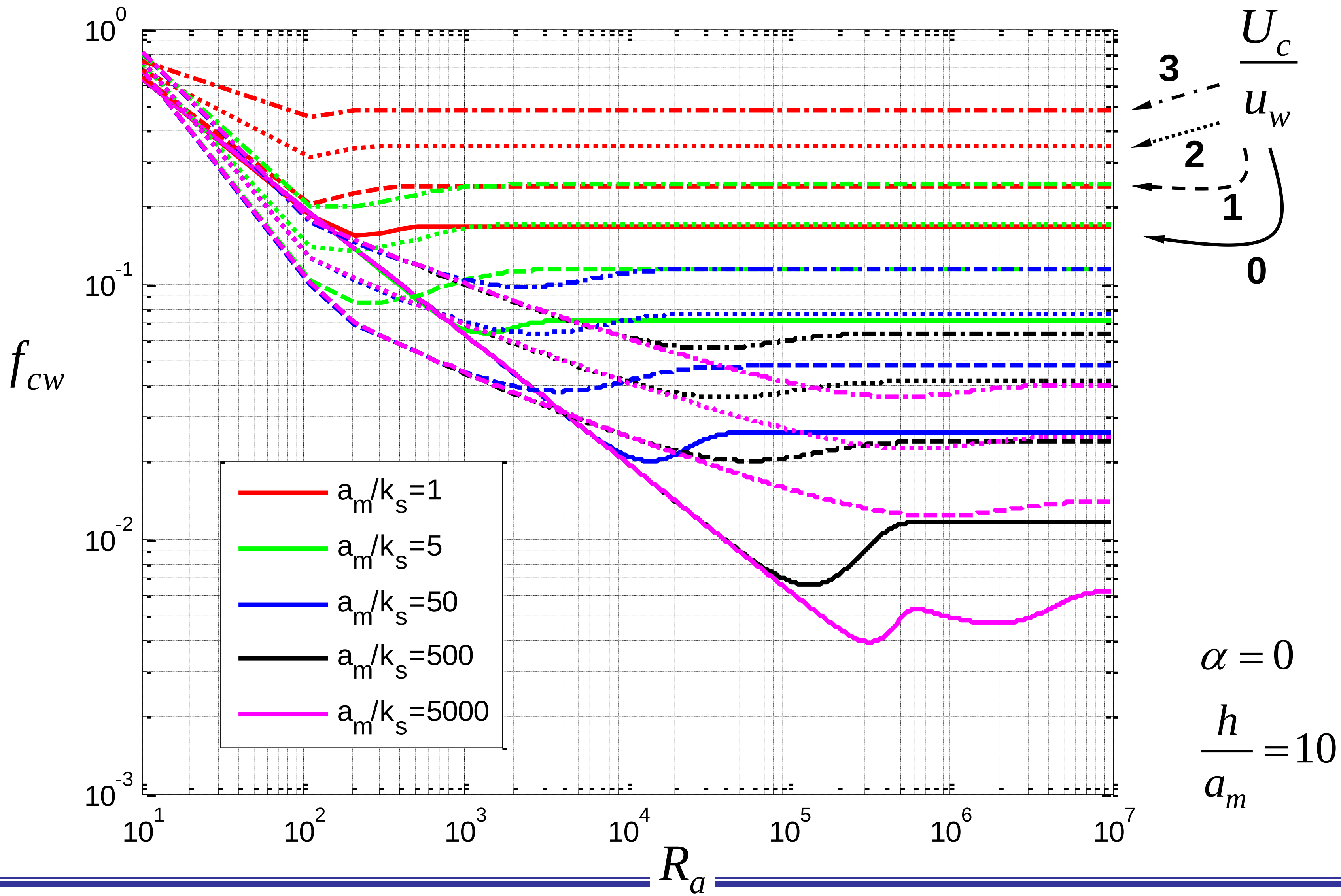
$$R_{cw} = 500R_c + R_a,\quad R_1 = 0.501\zeta^{1.15},\ \zeta = 350\gamma\frac{h}{z_0} + (1-\gamma)\frac{a_m}{z_0},\ \gamma = \frac{U_c/u_w}{1+U_c/u_w}$$

$$\varepsilon = f_2\left\{ f_1\varepsilon_{(L)} + (1-f_1)\varepsilon_{(S)} \right\} + (1-f_2)\varepsilon_{(R)}$$

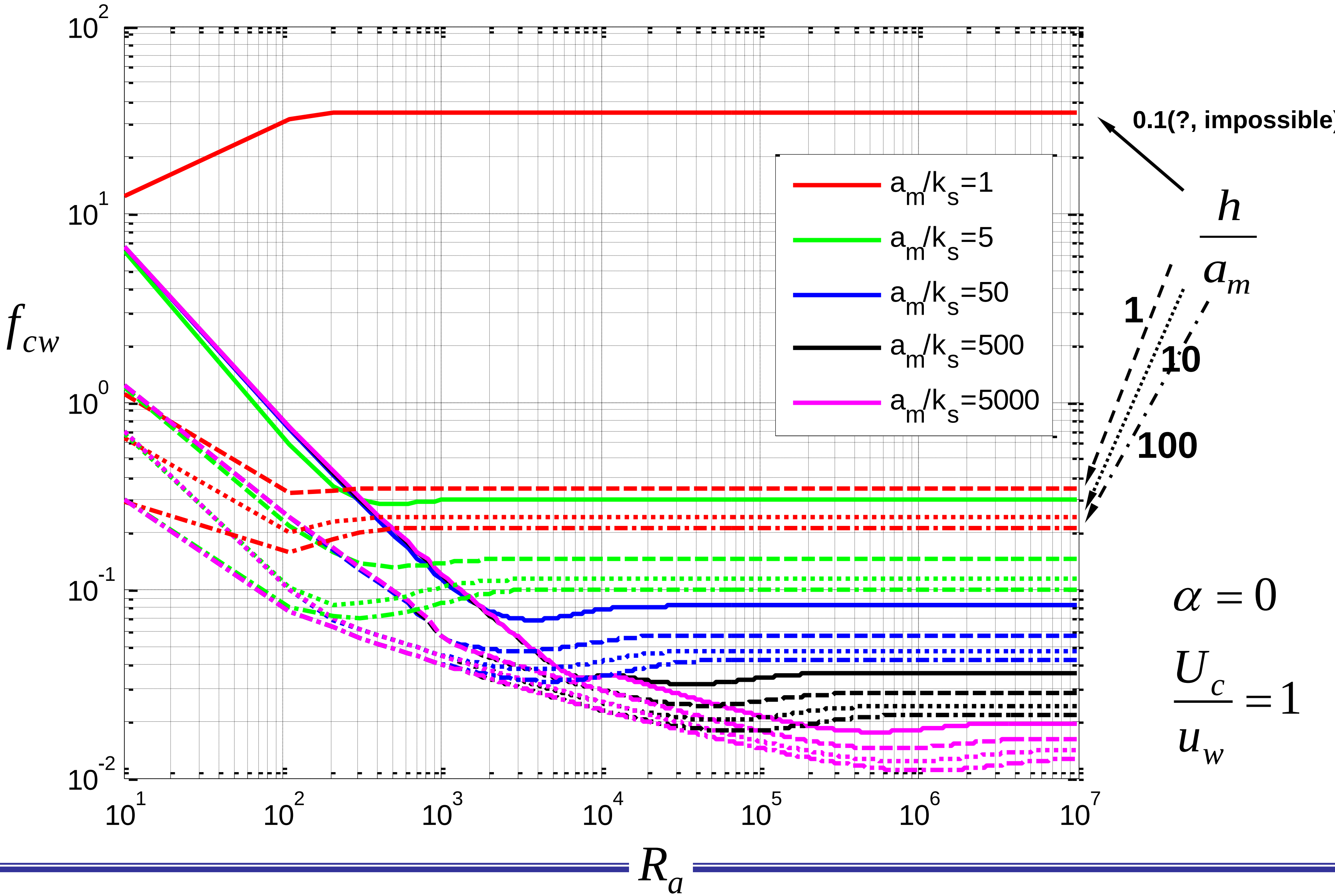
Transitions between flow regimes

Laminar to Turbulent $\quad 2.5\times10^5 < R_{cw} < 6.0\times10^5$

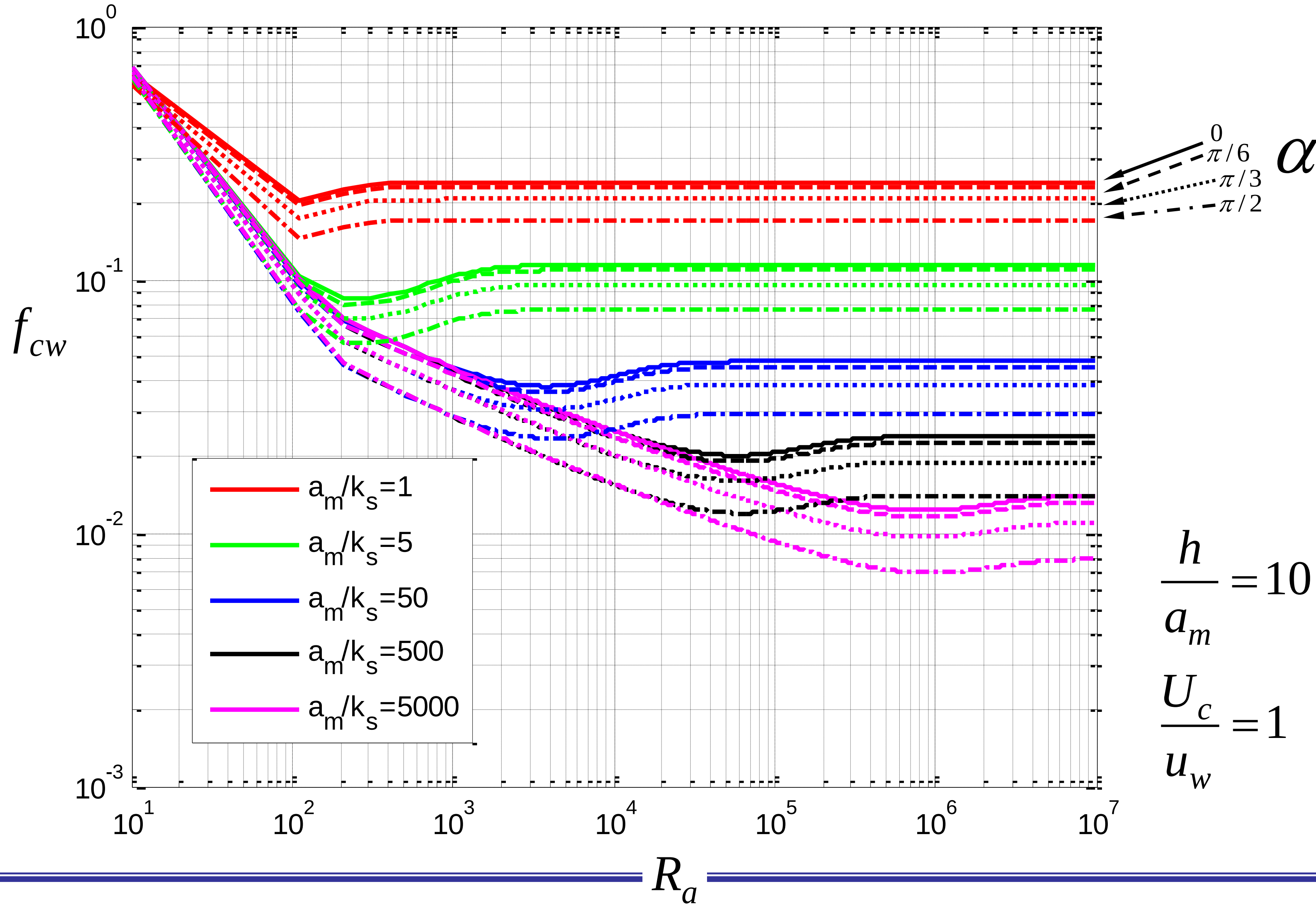
Smooth to Rough $\quad 0.501\zeta^{1.15} < R_{cw} < 7.00\zeta^{1.15}$



Wave-current friction factor for various U_c/u_w



Wave-current friction factor for various h/a_m



Wave-current friction factor for various *angle*

(3) Soulsby(1997)

Maximum bottom shear stress during a wave cycle

$$\tau_b = \sqrt{(\tau_m + \tau_w|\cos\lambda|)^2 + (\tau_w|\sin\lambda|)^2}$$

λ: the angle between wave and current

Mean bottom shear stress

$$\tau_m = \tau_c\left[1 + 1.2\left(\frac{\tau_w}{\tau_c + \tau_w}\right)^{3.2}\right]$$

$$\tau_w = \frac{1}{2}\rho_w f_w U_w^2. \qquad \tau_c = \rho u_*^2 \qquad \frac{u}{u_*} = \frac{1}{\kappa}\ln\left(\frac{z}{z_o}\right)$$

$$f_w = \max\{f_{wr}, f_{ws}\},\ f_{wr} = 0.237 r^{-0.52},\ f_{ws} = B\,\mathrm{Re}_w^{-N},$$

$$r = A/k_s, \mathrm{Re}_w = U_w A/\nu, A = U_w T/(2\pi), U_w = \frac{\pi H}{T\sinh(2\pi d/L)}$$

$$B = 2, N = 0.5 \text{ for } \mathrm{Re}_w \le 5\times10^5; B = 0.0521, N = 0.187 \text{ for } \mathrm{Re}_w > 5\times10^5$$

A : maximum bottom wave orbital amplitude

U_w: maximum bottom wave orbital velocity

In a Coexistent Wave-current Field

- 如果作用于光滑床面上潮流和波浪的速度够小且水流为层流，则床面切应力的合力是潮流与波浪各自所产生的切应力的简单线性叠加；
- 若潮流与波浪强度较大，水流就会产生紊动，紊流产生在潮流与波浪的边界层内，则切应力的合力是非线性叠加的。

作业二

- At the nearshore of Zhoushan Island, wave and current that co-exist may lead to sediment resuspension. The flow can be assumed as turbulence over a rough bottom. The angle between the wave and current is 30° and water depth is 5 m. The current speed at z=0.1 m is 0.2 m/s, and k_s is 0.15 m. The wave height is 2 m and wave period is 5 s. Please estimate the maximum total bottom shear stress by using Kajiura (1968)'s explicit method, US CERC (1984), and Soulsby (1997) methods. The critical shear stress for the bottom sediment is 0.1 N/m^2. Can the wave and current induced bottom shear stress initiate the bottom sediments?

Due 2018/5/22 (in class).

Thank You